The Best and the Worst: Computing the Optimal Value Range in Interval Linear Programming

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Consider a linear programming problem...

$$\text{minimize } c^T x \text{ subject to } Ax \leq b$$

approximation and rounding

$b \approx 3.14159$

Consider a linear programming problem...

$$\text{minimize } c^T x \text{ subject to } Ax \leq b$$

estimating the future

€15.6 $\leq c \leq$ €17.1

representing missing data

$0.66,\ 0.21,\ 0.84,\ d = ?,\ 0.05$

inexact measurements

$a = 5 \pm 0.05g$

discretization of time

$t_{min} = 22$°C, $t_{max} = 23.5$°C

Interval Linear Programming

Consider an **interval** linear programming problem...

$$\text{minimize } [c]^T x \text{ subject to } [A]x \leq [b]$$

approximation and rounding

$[b] = [3.141592, 3.141593]$

estimating the future

$[c] = [15.6, 17.1]$

representing missing data

$[d] = [0, 1]$

inexact measurements

$[a] = [4.95, 5.05]$

discretization of time

$[t] = [22, 23.5]$

Interval Linear Programming: Definitions

- Given two real matrices $\underline{A}, \overline{A} \in \mathbb{R}^{m \times n}$ with $\underline{A} \leq \overline{A}$, we define an **interval matrix** as the set

$$[A] = [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m \times n} : \underline{A} \leq A \leq \overline{A}\}.$$

- An **interval linear program** is a family of linear programs

$$\begin{aligned} \text{minimize} \quad & a^T x + c^T y \\ \text{subject to} \quad & Ax + By = b, \\ & Cx + Dy \leq d, \\ & x \geq 0, \end{aligned}$$

  where $A \in [A], B \in [B], C \in [C], D \in [D], a \in [a], b \in [b], c \in [c], d \in [d]$.

- A linear program in the family is called a **scenario**.

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$[a]^T x = b$
is not
$[a]^T x \leq b, [a]^T x \geq b$

where $A \in [A], B \in [B], C \in [C], D \in [D], a \in [a], b \in [b], c \in [c]$, $d \in [d]$.

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Feasibility and Optimality

- A vector x is a **weakly feasible/optimal** solution to the interval program, if x is a feasible/optimal solution for *some* scenario.
- A vector x is a **strongly feasible/optimal** solution to the interval program, if x is a feasible/optimal solution for *each* scenario.
- Regarding optimal values, we usually consider the **best** and the **worst** optimal value (or the **optimal value range**)

$$\underline{f}([A],[b],[c]) = \inf\{f(A,b,c) : A \in [A], b \in [b], c \in [c]\},$$
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Interval Linear Programming: Example

$$\begin{aligned} \text{maximize} \quad & x_2 \\ \text{subject to} \quad & [-1, 1]x_1 + x_2 \leq 0 \\ & x_2 \leq 1 \end{aligned}$$

Interval Linear Programming: Example

Let's traverse through this!

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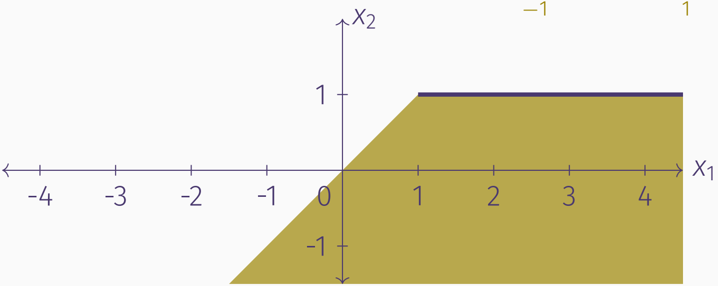

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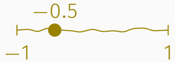

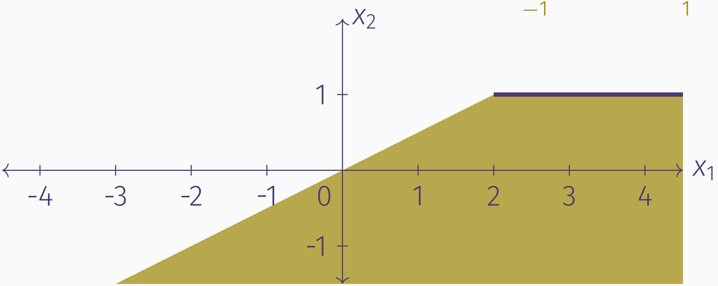

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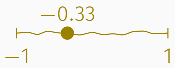

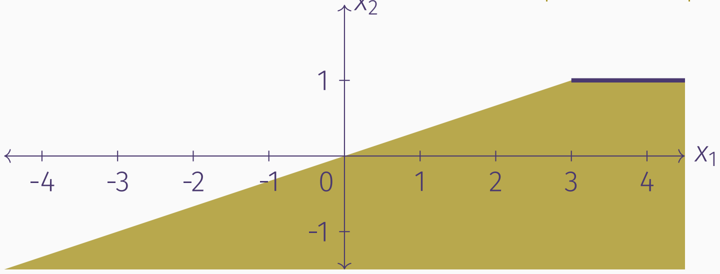

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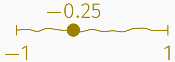

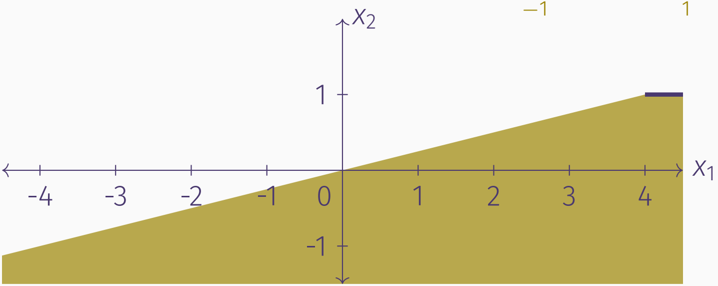

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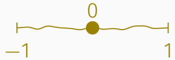

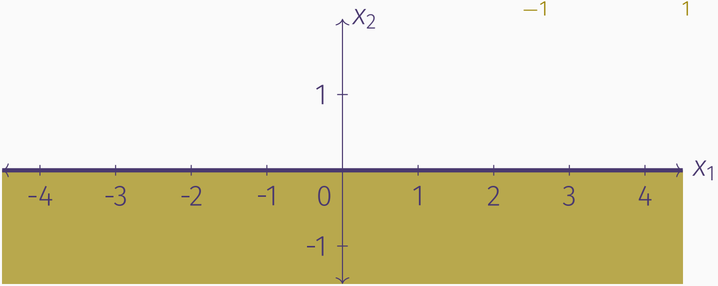

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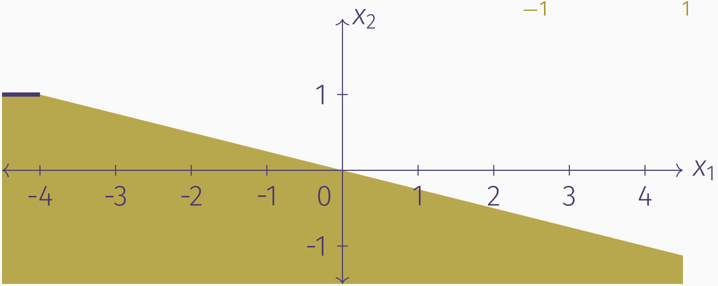

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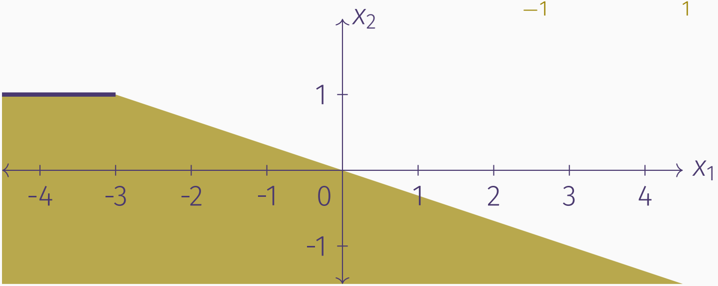

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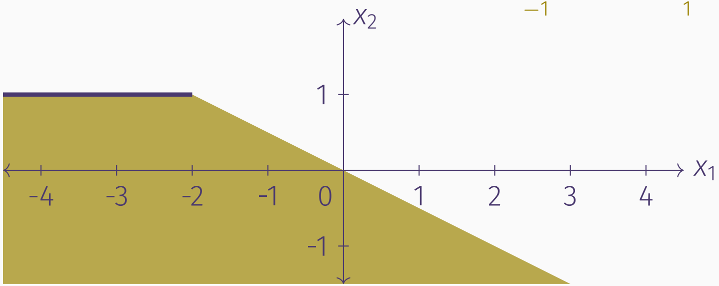

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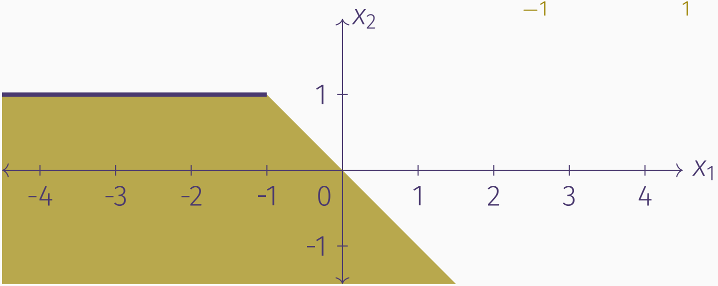

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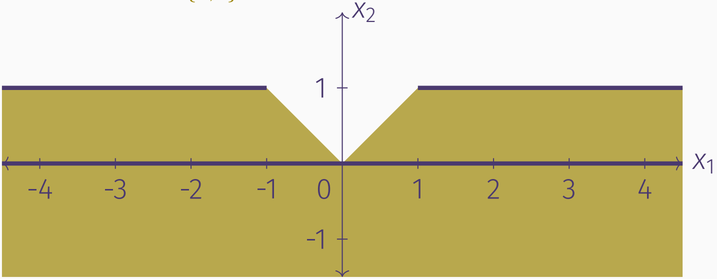
Optimal values: $\{0, 1\}$



Best optimal value:

$\underline{f} = \inf \underline{c}^T x : \underline{A}x \leq \overline{b}, \overline{A}x \geq \underline{b}, x \geq 0$

Theorem (Oettli, Prager, 1964): x solves $[A]x = [b] \Leftrightarrow |A_c x - b_c| \leq A_\Delta |x| + b_\Delta$

Worst optimal value:

$\overline{f} = \sup_{s \in \{\pm 1\}^m} f(A_c - \mathrm{diag}(s) A_\Delta, b_c + \mathrm{diag}(s) b_\Delta, \overline{c})$

Theorem (Rohn, 1997): Deciding whether $\overline{f}(A, [b], c) \geq 1$ holds is NP-hard for interval linear programs of type $\min c^T x : Ax = [b],\ x \geq 0$.

Since computing the worst optimal value $\overline{f}$ exactly is difficult, we can try to find an upper bound $\overline{f}^U$ and a lower bound $\overline{f}^L$ (e.g. iterative improvement from a scenario or relaxations).

Semi-strong Optimality

A vector $x \in \mathbb{R}^n$ is[1]...

- a $(\emptyset)$**-strong optimal solution** of the ILP if it is an optimal solution for some scenario with $A \in [A], b \in [b], c \in [c]$.
- a $([c])$**-strong optimal solution** of the ILP if for each $c \in [c]$ there exist $A \in [A], b \in [b]$ such that x is optimal for the scenario (A, b, c).
- a $([b])$**-strong optimal solution** of the ILP if for each $b \in [b]$ there exist $A \in [A], c \in [c]$ such that x is optimal for the scenario (A, b, c).
- ...
- a $([b], [c])$**-strong optimal solution** of the ILP if for each $b \in [b], c \in [c]$ there exists $A \in [A]$ such that x is optimal for the scenario (A, b, c).
- an $([A], [b], [c])$**-strong optimal solution** of the ILP if it is an optimal solution for each scenario with $A \in [A], b \in [b], c \in [c]$.

[1] Luo, J., Li, W., Strong optimal solutions of interval linear programming (2013).

From Optimal Values to Semi-strong Values

Let us now reformulate the problem of computing the optimal value range...

- A value $r \in \mathbb{R}$ is a **weak value**, if there is a scenario of the program with $f(A, b, c) \leq r$.
- A value $r \in \mathbb{R}$ is a **strong value**, if $f(A, b, c) \leq r$ holds for each scenario.

Then, the best and the worst optimal value can be viewed as the best of all weak or strong values, respectively.

Semi-strong Values

A value $r \in \mathbb{R}$ is...

a **($\emptyset$)-strong value** of the ILP if $f(A, b, c) \leq r$ holds for some scenario with $A \in [A], b \in [b], c \in [c]$.

a **($[c]$)-strong value** of the ILP if for each $c \in [c]$ there exist $A \in [A]$, $b \in [b]$ such that $f(A, b, c) \leq r$.

a **($[b]$)-strong value** of the ILP if for each $b \in [b]$ there exist $A \in [A]$, $c \in [c]$ such that $f(A, b, c) \leq r$.

...

a **($[b], [c]$)-strong value** of the ILP if for each $b \in [b], c \in [c]$ there exists $A \in [A]$ such that $f(A, b, c) \leq r$.

an **($[A], [b], [c]$)-strong value** of the ILP if $f(A, b, c) \leq r$ holds for each scenario with $A \in [A], b \in [b], c \in [c]$.

Testing Semi-strong Values

Theorem

For each objective vector $c \in [c]$ there exist $A \in [A], b \in [b]$ with $f(A, b, c) \le r$ if and only if the interval linear system

$$[c]^T x \le r,$$
$$x \text{ is weakly feasible}$$

is strongly feasible.

An interval linear system

$$[A]x + [B]y = [b],\ [C]x + [D]y \le [d],\ x \ge 0$$

is strongly feasible if and only if the linear system

$$\begin{aligned}(A_c + T_p A_\Delta)x + (B_c + T_p B_\Delta)y^1 - (B_c - T_p B_\Delta)y^2 &= b_c - T_p b_\Delta,\\ \overline{C}x + \overline{D}y^1 - \underline{D}y^2 &\le \underline{d},\\ x, y^1, y^2 &\ge 0\end{aligned}$$

is feasible for each $p \in \{\pm 1\}^k$.

Testing Semi-strong Values

Theorem

For each objective vector $c \in [c]$ there exist $A \in [A], b \in [b]$ with $f(A, b, c) \leq r$ if and only if the interval linear system

$$[c]^T x \leq r,$$
$$\underline{A}x \leq \overline{b}, -\overline{A}x \leq -\underline{b}, x \geq 0$$

is strongly feasible.

Theorem (Oettli, Prager, 1964):

x solves $[A]x = [b] \;\Leftrightarrow\; |A_c x - b_c| \leq A_\Delta |x| + b_\Delta$

Testing Semi-strong Values (cont.)

Theorem

For each right-hand side $b \in [b]$ there exists a constraint matrix $A \in [A]$ and an objective vector $c \in [c]$ with $f(A, b, c) \leq r$ if and only if the system

$$\underline{c}^T x \leq r, \quad \underline{A} x \leq z, \quad -\overline{A} x \leq -z, \quad x \geq 0, \quad z = [b]$$

is strongly feasible.

Theorem

For each $A \in [A]$ there exists an objective vector $c \in [c]$ and a right-hand-side vectors $b \in [b]$ with $f(A, b, c) \leq r$ if and only if the system

$$\underline{c}^T x \leq r, \quad [A] x = z, \quad \underline{b} \leq z \leq \overline{b}, \quad x \geq 0$$

is strongly feasible.

Conclusion

- For interval linear programs, we usually compute the best and the worst possible optimal values (the optimal value range), which can also be interpreted in the context of weak and strong properties.
- We introduced semi-strong values that can serve as a generalization of the optimal value range, based on generalized concepts of feasibility and optimality.
- Conditions for testing semi-strong values can be formulated in terms of weak and strong feasibility.

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Thank you for your attention!