On the Properties of Interval Linear Programs with a Fixed Coefficient Matrix

Elif Garajová[1] (with Milan Hladík[1] & Miroslav Rada[2])

[1]Department of Applied Mathematics, Faculty of Mathematics and Physics, Charles University

[2]Faculty of Informatics and Statistics, University of Economics, Prague

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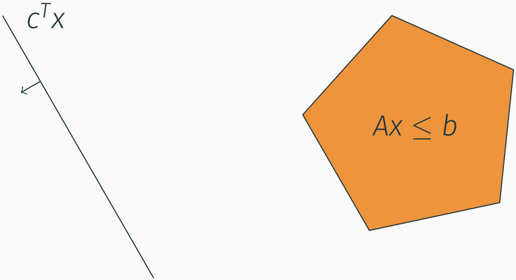
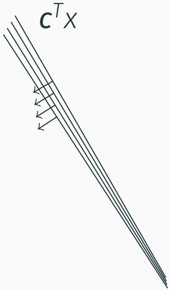
minimize $c^T x$ subject to $Ax \leq b$

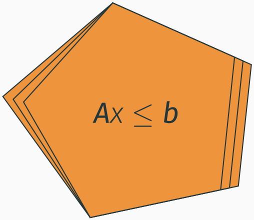


Interval Linear Programming

$$\text{minimize } \boldsymbol{c}^T x \text{ subject to } \boldsymbol{A}x \leq \boldsymbol{b}$$

interval coefficients





Feasible and Optimal Solutions

Formally…

- An interval linear program is a family of linear programs:

$$\text{minimize } c^T x \text{ subject to } Ax \leq b \text{ with } A \in \boldsymbol{A}, b \in \boldsymbol{b}, c \in \boldsymbol{c},$$

- x^* is (weakly) feasible, if $Ax^* \leq b$ for some $A \in \boldsymbol{A}, b \in \boldsymbol{b}$,
- x^* is (weakly) optimal, if it is an optimal solution of a linear program $\min c^T x : Ax \leq b$ with $A \in \boldsymbol{A}, b \in \boldsymbol{b}, c \in \boldsymbol{c}$.

Dependency Problem

$$\begin{aligned} \max \quad & x_1 \\ \text{s. t.} \quad & [0,1]x_1 - x_2 = 0, \\ & x_2 \leq 1, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Optimal set: $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in [1, \infty) \text{ and } x_2 = 1\}$

Dependency Problem

$$
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Dependency Problem

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$$\begin{aligned} \max \quad & x_1 \\ \text{s. t.} \quad & 1x_1 - x_2 \leq 0, \\ & 0x_1 - x_2 \geq 0, \\ & x_2 \leq 1, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Optimal set: $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in [1, \infty) \text{ and } x_2 = 1\}$

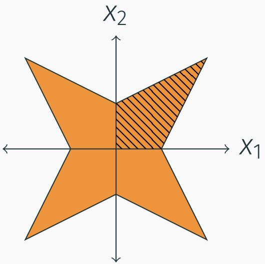
The solution $(0, 0)$ is now optimal, too!

Orthant Decomposition

Oettli–Prager (1964), Gerlach (1981)

$$x \in \mathbb{R}^n \text{ solves } \boldsymbol{A}x = \boldsymbol{b} \iff |A_c x - b_c| \leq A_\Delta |x| + b_\Delta$$

$$x \in \mathbb{R}^n \text{ solves } \boldsymbol{A}x \leq \boldsymbol{b} \iff A_c x - A_\Delta |x| \leq \overline{b}$$



Main Questions

Feasibility

- Are all/some scenarios feasible?
- Are there any (weakly) feasible solutions?

(Un)boundedness

- Do all/some scenarios have an unbounded objective function?

Optimality

- Do all/some scenarios possess an optimal solution?
- Are there any (weakly) optimal solutions?

Computational Complexity

	$\min \boldsymbol{c}^T x$ $Ax = \boldsymbol{b},\ x \geq 0$	$\min \boldsymbol{c}^T x$ $Ax \leq \boldsymbol{b}$	$\min \boldsymbol{c}^T x$ $Ax \leq \boldsymbol{b},\ x \geq 0$
strong feasibility	co-NP-hard	polynomial	polynomial
weak feasibility	polynomial	NP-hard	polynomial
strong unboundedness	co-NP-hard	polynomial	polynomial
weak unboundedness	?	NP-hard	polynomial
strong optimality	co-NP-hard	co-NP-hard	polynomial
weak optimality	NP-hard	NP-hard	?

Computational Complexity

Let's fix the matrix A

	$\min c^T x$ $Ax = b,\ x \geq 0$	$\min c^T x$ $Ax \leq b$	$\min c^T x$ $Ax \leq b,\ x \geq 0$
strong feasibility	?	polynomial	polynomial
weak feasibility	polynomial	?	polynomial
strong unboundedness	?	polynomial	polynomial
weak unboundedness	?	?	polynomial
strong optimality	?	?	polynomial
weak optimality	?	?	?

Weak Feasibility

Oettli–Prager (1964), Gerlach (1981)

$$x \in \mathbb{R}^n \text{ solves } \boldsymbol{A}x = \boldsymbol{b} \;\Leftrightarrow\; |A_c x - b_c| \le A_\Delta |x| + b_\Delta$$

$$x \in \mathbb{R}^n \text{ solves } \boldsymbol{A}x \le \boldsymbol{b} \;\Leftrightarrow\; A_c x - A_\Delta |x| \le \overline{b}$$

In general, testing weak feasibility is NP-hard for type $\boldsymbol{A}x \le \boldsymbol{b}$.

Special case with a fixed matrix is polynomial:

1. $\underline{b} \le Ax \le \overline{b},\; x \ge 0,$
2. $Ax \le \overline{b},$
3. $Ax \le \overline{b},\; x \ge 0.$

Main Result

Theorem

Testing weak feasibility is NP-hard for interval systems in the form $Ax \leq 0,\ \boldsymbol{b}^T x < 0$.

Proof idea:

$e = (1, \dots, 1)^T$

1. Checking feasibility of the system $|Ax| \leq e,\ e^T|x| > 1$ is an NP-hard problem. Fact!
2. This problem is equivalent to checking feasibility of the system $|Ax| \leq ey,\ y \geq 0,\ e^T|x| > y$.
3. The inequality $e^T|x| - y > 0$ is feasible if and only if the interval inequality $[-e, e]^T x + y < 0$ is weakly feasible.
   (By the Gerlach Theorem)

Main Result (cont.)

$$|Ax| \leq e,\ e^T|x| > 1 \text{ is feasible} \tag{1}$$

$$\Updownarrow$$

$$|Ax| \leq ey,\ y \geq 0,\ e^T|x| > y \text{ is feasible} \tag{2}$$

- If x is a feasible solution of (1), then the pair $(x, 1)$ solves system (2).
- Let (x, y) be a solution of (2). If $y > 0$, then $\frac{x}{y}$ solves (1).
- Otherwise, we have $Ax = 0, e^T|x| > 0$ and $\frac{x}{e^T|x|-\varepsilon}$ for some ε with $0 < \varepsilon < e^T|x|$ solves (1).

Strong Feasibility

Theorem

Testing strong feasibility is co-NP-hard for interval linear systems of type $Ax = \mathbf{b}, x \geq 0$.

Why? $Ax = \mathbf{b}, x \geq 0$ is weakly infeasible

$$\Updownarrow$$

$A^T y \geq 0, \mathbf{b}^T y < 0$ is weakly feasible

Strong Feasibility

Theorem

Testing strong feasibility is co-NP-hard for interval linear systems of type $Ax = \mathbf{b}, x \geq 0$.

Why? $Ax = \mathbf{b}, x \geq 0$ is weakly infeasible

$\Updownarrow$ ← Farkas' Lemma

$A^T y \geq 0, \mathbf{b}^T y < 0$ is weakly feasible

NP-hard by the main theorem

Strong Feasibility: Testing

For equation-constrained programs, we need to check all extremal scenarios:

1. $Ax = b_c + \operatorname{diag}(p) b_\Delta,\ x \geq 0$ for each $p \in \{\pm 1\}^m$.

For inequalities, there is a worst-case scenario:

2. $Ax \leq \underline{b}$,
3. $Ax \leq \underline{b},\ x \geq 0$.

Weak Unboundedness

Theorem

Testing weak unboundedness is NP-hard for ILPs of type $\min \boldsymbol{c}^T x : Ax \leq \boldsymbol{b}$.

Why?

$$\min \boldsymbol{c}^T x : Ax \leq \boldsymbol{b} \text{ is weakly unbounded}$$

$$\Updownarrow$$

$$Ax \leq \overline{b},\ Ad \leq 0,\ \boldsymbol{c}^T d < 0 \text{ is weakly feasible}$$

Weak Unboundedness

Theorem

Testing weak unboundedness is NP-hard for ILPs of type $\min \boldsymbol{c}^T x : Ax \leq \boldsymbol{b}$.

Why?

$$\min \boldsymbol{c}^T x : Ax \leq \boldsymbol{b} \text{ is weakly unbounded}$$

$$\Updownarrow$$

feasibility + unboundedness

$$Ax \leq \overline{b},\ Ad \leq 0,\ \boldsymbol{c}^T d < 0 \text{ is weakly feasible}$$

NP-hard by the main theorem

Weak Unboundedness: Testing

We need to test weak feasibility of the original problem and the unboundedness constraints:

1. $\underline{b} \leq Ax \leq \overline{b},\ x \geq 0,\ Ad = 0,\ d \geq 0,\ \underline{c}^T d \leq -1,$
2. $Ax \leq \overline{b},\ Ad \leq 0,\ (c_c^T - c_\Delta^T \mathrm{diag}(p))d \leq -1$
   for some $p \in \{\pm 1\}^n$,
3. $Ax \leq \overline{b},\ x \geq 0,\ Ad \leq 0,\ d \geq 0,\ \underline{c}^T d \leq -1.$

Strong Unboundedness

Theorem

Testing strong unboundedness is co-NP-hard for ILPs of type $\min \boldsymbol{c}^T x : Ax = \boldsymbol{b}, x \geq 0$.

Why?

$$\text{maximize } z \text{ subject to } Ax = \boldsymbol{b},\ x \geq 0,\ z \geq 0$$

is strongly unbounded

$$\Updownarrow$$

$Ax = \boldsymbol{b},\ x \geq 0$ is strongly feasible

Proved to be co-NP-hard

Strong Unboundedness: Testing

We want to test unboundedness in the worst-case scenario:

1. minimize $\overline{c}^T x$ subject to $Ax = b_c + \operatorname{diag}(p) b_\Delta,\ x \geq 0$ for each $p \in \{\pm 1\}^m$,
2. minimize $\overline{c}^T x^1 - \underline{c}^T x^2$ subject to $A(x^1 - x^2) \leq \underline{b},\ x^1 \geq 0,$ $x^2 \geq 0,$
3. minimize $\overline{c}^T x$ subject to $Ax \leq \underline{b},\ x \geq 0.$

Weak Optimality

For testing weak optimality, we only need to test weak feasibility of the primal and the dual problem:

1. $\underline{b} \leq Ax \leq \overline{b},\ x \geq 0,\ A^T y \leq \overline{c},$
2. $Ax \leq \overline{b},\ \underline{c} \leq A^T y \leq \overline{c},\ y \leq 0,$
3. $Ax \leq \overline{b},\ x \geq 0,\ A^T y \leq \overline{c},\ y \leq 0.$

Note: This is not sufficient in the general case!

Strong Optimality

Theorem

Testing strong optimality is co-NP-hard for ILPs of types $\min \boldsymbol{c}^T x : Ax = \boldsymbol{b}, x \geq 0$ and $\min \boldsymbol{c}^T x : Ax \leq \boldsymbol{b}$.

Why?

$$\text{minimize } 0^T x \text{ subject to } Ax = \boldsymbol{b},\ x \geq 0 \text{ is strongly optimal}$$

$$\Updownarrow$$

$$Ax = \boldsymbol{b},\ x \geq 0 \text{ is strongly feasible}$$

Proved to be co-NP-hard

Strong Optimality: Testing

We can employ duality in linear programming and test strong feasibility of the primal and the dual problem:

1. $Ax = b_c + \mathrm{diag}(p) b_\Delta,\ x \geq 0,\ A^T y \leq \underline{c}$ for each $p \in \{\pm 1\}^m$,
2. $Ax \leq \underline{b},\ A^T y = c_c + \mathrm{diag}(p) c_\Delta,\ y \leq 0$ for each $p \in \{\pm 1\}^n$,
3. $Ax \leq \underline{b},\ x \geq 0,\ A^T y \leq \underline{c},\ y \leq 0.$

Complexity Results

	$\min \mathbf{c}^T x$ $Ax = \mathbf{b},\ x \geq 0$	$\min \mathbf{c}^T x$ $Ax \leq \mathbf{b}$	$\min \mathbf{c}^T x$ $Ax \leq \mathbf{b},\ x \geq 0$
strong feasibility	co-NP-hard	polynomial	polynomial
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Complexity Results

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weak optimality	polynomial	polynomial	polynomial

Thanks for your attention!