Seeking Optimality in Interval Linear Programming

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Interval Linear Programming

- An interval linear program is a family of linear programs

$$\text{minimize } c^T x \text{ subject to } Ax = b,\ x \geq 0$$

  where $A \in [A], b \in [b], c \in [c]$.

- A linear program in the family is called a scenario.

- Dependency problem:
  - $[A]x = [b] \ \rightarrow\ [A]x \leq [b],\ [A]x \geq [b]$
  - $[A]x \leq [b] \ \rightarrow\ [A]x^+ - [A]x^- \leq [b],\ x^+, x^- \geq 0$

The Questions

- What are the feasible solutions?
- What is the set of optimal solutions and values?
- Is a given solution feasible?
- Is a given feasible solution also optimal?
- Is the interval linear program bounded?
- ...

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But how do we define feasibility, optimality and other properties?

Seeking Optimal Values

Optimal value of an LP: $f(A, b, c) = \inf\{c^T x : Ax \leq b\}$

- $f(A, b, c) = -\infty$ if it is unbounded,
- $f(A, b, c) = \infty$ if it is infeasible,
- $f(A, b, c) = c^T x^*$ if there is an optimal solution x^*.

Optimal value range of an ILP:

- Lower bound of the optimal value range:

$$\underline{f}([A], [b], [c]) = \inf\{f(A, b, c) : A \in [A], b \in [b], c \in [c]\}$$

- Upper bound of the optimal value range:

$$\overline{f}([A], [b], [c]) = \sup\{f(A, b, c) : A \in [A], b \in [b], c \in [c]\}$$

Other concepts: Set of optimal values, Duality gap, . . .

Optimal Value Range

How to compute the optimal value range $[\underline{f}, \overline{f}]$?

Best optimal value:

$$\underline{f} = \inf \underline{c}^T x : \underline{A}x \leq \overline{b}, \overline{A}x \geq \underline{b}, x \geq 0$$

Worst optimal value:

$$\overline{f} = \sup_{s \in \{\pm 1\}^m} f(A_c - \operatorname{diag}(s) A_\Delta, b_c + \operatorname{diag}(s) b_\Delta, \overline{c})$$

Theorem (Rohn, 1997)

Deciding whether $\overline{f}(A, [b], c) \geq 1$ holds is NP-hard for interval linear programs of type $\min c^T x : Ax = [b], x \geq 0$.

Weak and Strong Properties

- We can study, whether a given property holds for at least one scenario of the program (weak property), or whether it holds for all scenarios (strong property).
- A given vector x is a weakly/strongly feasible solution to an interval linear program, if x is a feasible solution for some/each scenario with $A \in [A]$, $b \in [b], c \in [c]$.
- An interval linear program is weakly/strongly feasible, if some/each scenario of the program is feasible.

Weak and Strong Feasibility

Theorem (Oettli & Prager, 1964; Gerlach, 1981)

The interval linear system $[A]x = [b]$ *is weakly feasible*
$\Leftrightarrow$ $|A_c x - b_c| \leq A_\Delta |x| + b_\Delta$ *is feasible.*

The interval linear system $[A]x \leq [b]$ *is weakly feasible*
$\Leftrightarrow$ $A_c x - A_\Delta |x| \leq \overline{b}$ *is feasible.*

Theorem (Rohn, 1981; Rohn & Kreslová, 1994)

The interval linear system $[A]x = [b]$ *is strongly feasible*
$\Leftrightarrow$ $(A_c - \mathrm{diag}(s)A_\Delta)x_1 - (A_c + \mathrm{diag}(s)A_\Delta)x_2 = b_c - \mathrm{diag}(s)b_\Delta$, $x_1, x_2 \geq 0$ *is feasible for each* $s \in \{\pm 1\}^m$.

The interval linear system $[A]x \leq [b]$ *is strongly feasible*
$\Leftrightarrow$ $\overline{A}x_1 - \underline{A}x_2 \leq \underline{b}, x_1, x_2 \geq 0$ *is feasible.*

Weak and Strong Optimality of a Solution

A given vector x is a weakly/strongly optimal solution to an interval linear program, if x is an optimal solution for some/each scenario with $A \in [A]$, $b \in [b], c \in [c]$.

We have conditions for testing weak and strong optimality of a solution:

- M. Rada, M. Hladík, E. Garajová, Testing weak optimality of a given solution in interval linear programming revisited (2018).
- J. Luo, W. Li, Strong optimal solutions of interval linear programming (2013).

However, some of the cases are NP-hard to decide.

Describing the Weakly Optimal Set

Computing the interval hull of the set of all weakly optimal solutions is an NP-hard problem, in general.

- Linear programming algorithms
  - Interval simplex method (Machost, 1970; Gunn and Anders, 1981; Jansson, 1988; ...)
- Relaxations
  - Interval relaxation and orthant decomposition
  - Linearization of absolute value
- Parametric programming methods, Branch-and-bound
- Solving special cases
  - Linear programs with interval objective or right-hand side
  - Fixed coefficient matrix

Weak and Strong Optimality of a Program

An interval linear program is weakly/strongly optimal, if some/each scenario of the program has an optimal solution.

Theorem

An interval linear program $\min [c]^T x : [A]x = [b],\ x \geq 0$ *is weakly optimal if and only if the parametric program*

$$Ax = b,\ x \geq 0,\ A^T y \leq c,\ A \in [A], b \in [b], c \in [c]$$

is feasible.

Theorem

An interval linear program is strongly optimal if and only if it is strongly feasible and its dual program is also strongly feasible.

The Complexity of Weak Optimality (ILP)

Theorem

Testing weak optimality is NP-hard for all three basic types of interval linear programs.

Why?

1. $\min 0^T x : [A]x \leq [b]$ is weakly optimal
   $\Leftrightarrow [A]x \leq [b]$ is weakly feasible ← Proved to be NP-hard

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   $\Leftrightarrow \max [b]^T y : [A]^T y \leq [c]$ is weakly optimal

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2. $\min [c]^T x : [A]x = [b], x \geq 0$ is weakly optimal
   $\Leftrightarrow \max [b]^T y : [A]^T y \leq [c]$ is weakly optimal
3. We omit the proof for $\min [c]^T x : [A]x \leq [b], x \geq 0$.

Theorem

Testing strong optimality is co-NP-hard for interval programs of types $\min [c]^T x : [A]x = [b], x \geq 0$ and $\min [c]^T x : [A]x \leq [b]$.

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2. $\min [c]^T x : [A]x \leq [b]$ is strongly optimal
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Why?

1. $\min 0^T x : [A]x = [b],\ x \geq 0$ is strongly optimal
   $\Leftrightarrow [A]x = [b],\ x \geq 0$ is strongly feasible
2. $\min [c]^T x : [A]x \leq [b]$ is strongly optimal
   $\Leftrightarrow \max [b]^T y : [A]^T y = [c], y \leq 0$ is strongly optimal
3. $\min [c]^T x : [A]x \leq [b], x \geq 0$ is strongly optimal
   $\Leftrightarrow \overline{A}x \leq \underline{b}, x \geq 0, \underline{A}^T y \leq \underline{c}, y \leq 0$ is feasible

Overview of Complexity

		$\min [c]^T x$ $[A]x = [b],\ x \geq 0$	$\min [c]^T x$ $[A]x \leq [b]$	$\min [c]^T x$ $[A]x \leq [b],\ x \geq 0$
of a program	strong feasibility	co-NP-hard	polynomial	polynomial
	weak feasibility	polynomial	NP-hard	polynomial
	strong optimality	co-NP-hard	co-NP-hard	polynomial
	weak optimality	NP-hard	NP-hard	NP-hard
of a solution	strong feasibility	polynomial	polynomial	polynomial
	weak feasibility	polynomial	polynomial	polynomial
	strong optimality	?	co-NP-hard	?
	weak optimality	NP-hard	polynomial	polynomial

Basis Stability

Definition

Given a basis $B \subseteq \{1, \ldots, n\}$, an interval linear program

$$\text{minimize } [c]^T x \text{ subject to } [A]x = [b],\ x \geq 0$$

is B-stable, if B is an optimal basis for each scenario.

Theorem

Under unique B-stability, the set of all weakly optimal solutions is

$$\underline{A}_B x_B \leq \overline{b},\ -\overline{A}_B x_B \leq -\underline{b},\ x_B \geq 0,\ x_N = 0.$$

Other Concepts of Feasibility

- A vector $x \in \mathbb{R}^n$ is a tolerance solution of $[A]x = [b]$ if for each $A \in [A]$ there exists a $b \in [b]$ such that $Ax = b$ holds.

- A vector $x \in \mathbb{R}^n$ is a control solution of $[A]x = [b]$ if for each $b \in [b]$ there exists an $A \in [A]$ such that $Ax = b$ holds.

- Split the coefficients to universally and existentially quantified: Let $[A] = [A^\forall] + [A^\exists],\ [b] = [b^\forall] + [b^\exists]$. A vector $x \in \mathbb{R}^n$ is an AE solution of $[A]x = [b]$ if

$$(\forall A^\forall \in [A^\forall])(\forall b^\forall \in [b^\forall])(\exists A^\exists \in [A^\exists])(\exists b^\exists \in [b^\exists]) :$$
$$(A^\forall + A^\exists)x = b^\forall + b^\exists.$$

Generalized Strong Optimality

A vector $x \in \mathbb{R}^n$ is[1]...

- a $(\emptyset)$-strong optimal solution of the ILP if it is an optimal solution for some scenario with $A \in [A], b \in [b], c \in [c]$.
- a $([c])$-strong optimal solution of the ILP if for each $c \in [c]$ there exist $A \in [A], b \in [b]$ such that x is optimal for the scenario (A, b, c).
- a $([b])$-strong optimal solution of the ILP if for each $b \in [b]$ there exist $A \in [A], c \in [c]$ such that x is optimal for the scenario (A, b, c).
- ...
- a $([b], [c])$-strong optimal solution of the ILP if for each $b \in [b], c \in [c]$ there exists $A \in [A]$ such that x is optimal for the scenario (A, b, c).
- an $([A], [b], [c])$-strong optimal solution of the ILP if it is an optimal solution for each scenario with $A \in [A], b \in [b], c \in [c]$.

[1] Luo, J., Li, W., Strong optimal solutions of interval linear programming (2013).

Generalized Optimality of a Program

Theorem

An interval linear program $\min\, [c]^T x : [A]x = [b],\ x \geq 0$ *is* (A)*-strongly optimal if and only if the interval linear system*

$$[A]x = b, x \geq 0, \underline{b} \leq b \leq \overline{b},$$
$$[A]^T y \leq c, \underline{c} \leq c \leq \overline{c}$$

is strongly feasible.

An analogous result can be obtained for (A, b)-strong and (A, c)-strong optimality of an ILP.

Even More Generalization

Generalized strong optimality considers only $\forall\exists$-quantified definitions. By changing the order of the quantifiers, we can introduce even further notions of optimality and feasibility[2]:

- Is there a $c \in [c]$ such that the scenario (A, b, c) has an optimal solution for each $A \in [A], b \in [b]$?
- Is there a $c \in [c]$ such that for each $A \in [A]$ there is a $b \in [b]$ such that the scenario (A, b, c) has an optimal solution?

[2] Shary, S.P., A New Technique in Systems Analysis Under Interval Uncertainty and Ambiguity (2002).

Even More Generalized Optimality of a Program

Theorem

There is a $c \in [c]$ *such that the scenario* (A, b, c) *has an optimal solution for each* $A \in [A], b \in [b]$ *if and only if the interval linear system*

$$[A]x = [b], x \geq 0,$$
$$[A]^T y \leq \overline{c}$$

is strongly feasible.

Further Research

- Fast algorithms for tight enclosures of the optimal sets with respect to the various concepts of optimality.
- A unified systematic description of conditions for testing generalized strong optimality.
- Other properties of interval programs (boundedness, optimal values, etc.) in the generalized strong sense.
- Exploring a weaker notion of basis stability.

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Thanks for your attention!