The Effects of Transformations on the Optimal Set in Interval Linear Programming

Elif Garajová[1] (with Milan Hladík[1] & Miroslav Rada[2])

[1]Department of Applied Mathematics, Faculty of Mathematics and Physics, Charles University

[2]Faculty of Informatics and Statistics, University of Economics, Prague

The 14th International Symposium on Operations Research in Slovenia (SOR'17), Bled

Consider a linear programming problem...

$$\text{minimize } c^T x \text{ subject to } Ax \leq b$$

approximation and rounding

$b \approx 3.14159$

Consider a linear programming problem...

$$\text{minimize } c^T x \text{ subject to } Ax \leq b$$

estimating the future

€25.6 $\leq c \leq$ €27.1

inexact measurements

$a = 5 \pm 0.05g$

Interval Linear Programming

approximation and rounding

$\boldsymbol{b} = [3.141592, 3.141593]$

Consider an *interval* linear programming problem...

$$\text{minimize } \boldsymbol{c}^T x \text{ subject to } \boldsymbol{A}x \leq \boldsymbol{b}$$

estimating the future

$\boldsymbol{c} = [25.6, 27.1]$

inexact measurements

$\boldsymbol{a} = [4.95, 5.05]$

Interval Linear Programming: Definitions

- An interval linear program is a family of linear programs

$$\text{minimize } c^T x \text{ subject to } x \in \mathcal{M}(A, b),$$

  where $A \in \boldsymbol{A}, b \in \boldsymbol{b}, c \in \boldsymbol{c}$ and $\mathcal{M}(A, b)$ is the feasible set.
- A linear program in the family is called a scenario.
- A vector x is a (weakly) feasible/optimal solution to the interval program, if x is a feasible/optimal solution for some scenario with $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}, c \in \boldsymbol{c}$.

Interval Linear Programming: Example

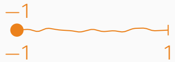
$$\begin{aligned} \text{maximize} \quad & x_2 \\ \text{subject to} \quad & [-1, 1]x_1 + x_2 \le 0 \\ & x_2 \le 1 \end{aligned}$$

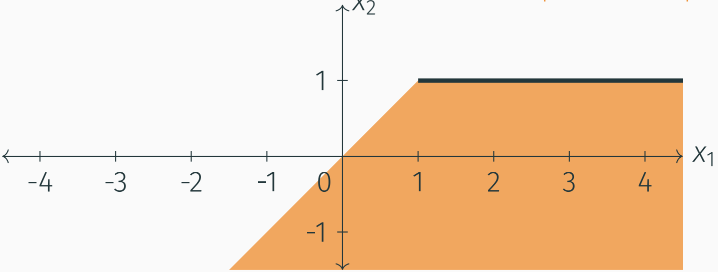
- What are the possible feasible solutions?
- Which solutions are optimal for some scenario?
- What is the set/range of all optimal values?

Interval Linear Programming: Example

Let's traverse through this!

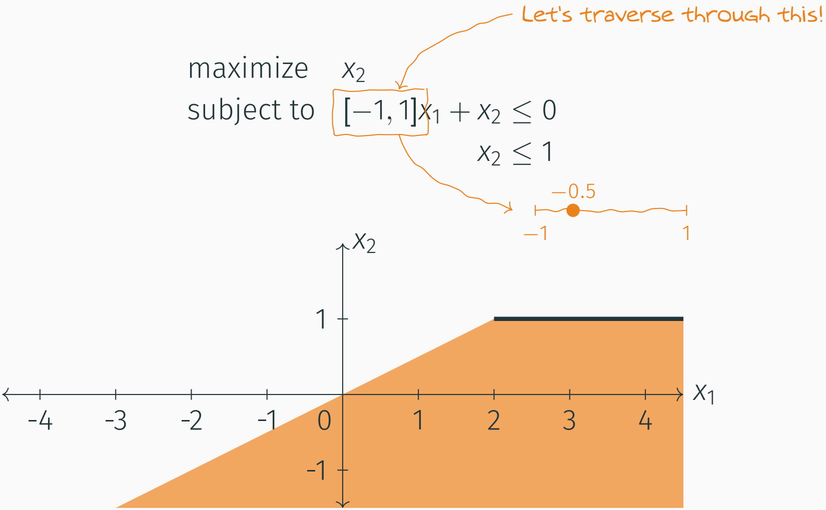
$$\begin{aligned} \text{maximize} \quad & x_2 \\ \text{subject to} \quad & [-1, 1]x_1 + x_2 \leq 0 \\ & x_2 \leq 1 \end{aligned}$$





Interval Linear Programming: Example

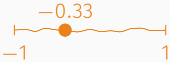
$$\begin{aligned} \text{maximize} \quad & x_2 \\ \text{subject to} \quad & [-1, 1]x_1 + x_2 \leq 0 \\ & x_2 \leq 1 \end{aligned}$$

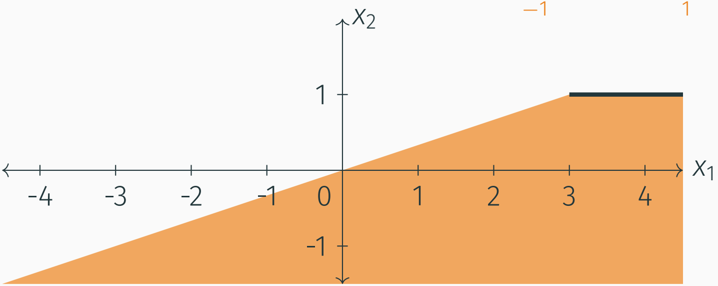


Interval Linear Programming: Example

Let's traverse through this!

$$\begin{aligned} \text{maximize} \quad & x_2 \\ \text{subject to} \quad & [-1, 1]x_1 + x_2 \leq 0 \\ & x_2 \leq 1 \end{aligned}$$

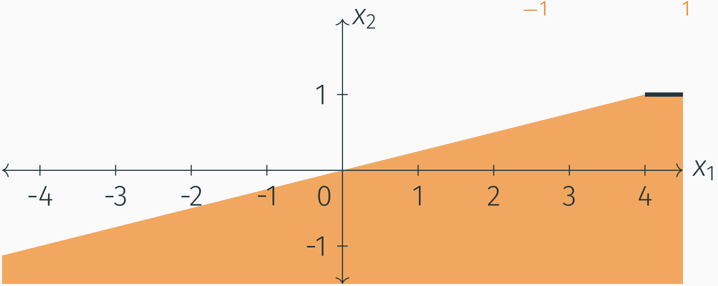




Interval Linear Programming: Example

Let's traverse through this!

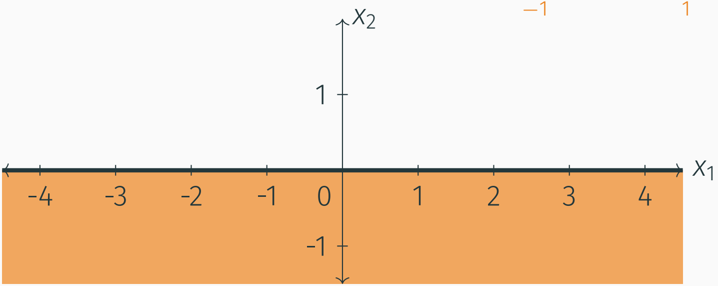
$$\begin{aligned} \text{maximize} \quad & x_2 \\ \text{subject to} \quad & [-1, 1]x_1 + x_2 \leq 0 \\ & x_2 \leq 1 \end{aligned}$$



Interval Linear Programming: Example

$$\begin{aligned} \text{maximize} \quad & x_2 \\ \text{subject to} \quad & [-1, 1]x_1 + x_2 \leq 0 \\ & x_2 \leq 1 \end{aligned}$$

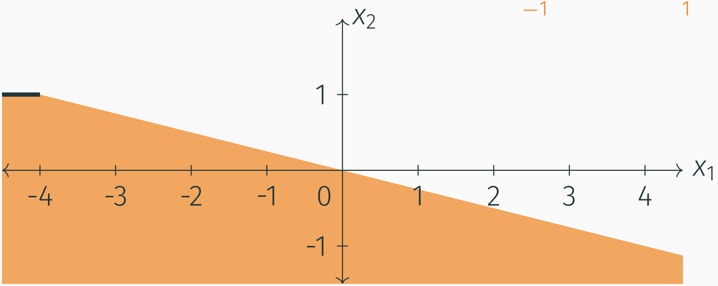
Let's traverse through this!



Interval Linear Programming: Example

Let's traverse through this!

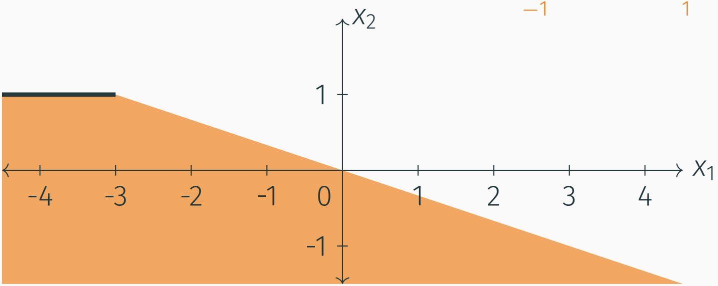
$$\begin{aligned} \text{maximize} \quad & x_2 \\ \text{subject to} \quad & [-1, 1]x_1 + x_2 \leq 0 \\ & x_2 \leq 1 \end{aligned}$$



Interval Linear Programming: Example

Let's traverse through this!

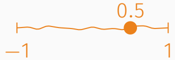
$$\begin{aligned} \text{maximize} \quad & x_2 \\ \text{subject to} \quad & [-1, 1]x_1 + x_2 \leq 0 \\ & x_2 \leq 1 \end{aligned}$$

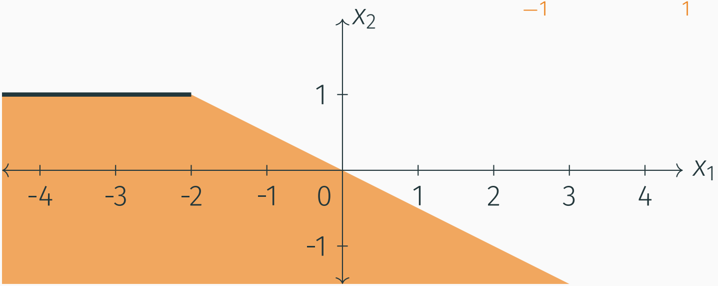


Interval Linear Programming: Example

Let's traverse through this!

$$\begin{aligned} \text{maximize} \quad & x_2 \\ \text{subject to} \quad & [-1, 1]x_1 + x_2 \leq 0 \\ & x_2 \leq 1 \end{aligned}$$

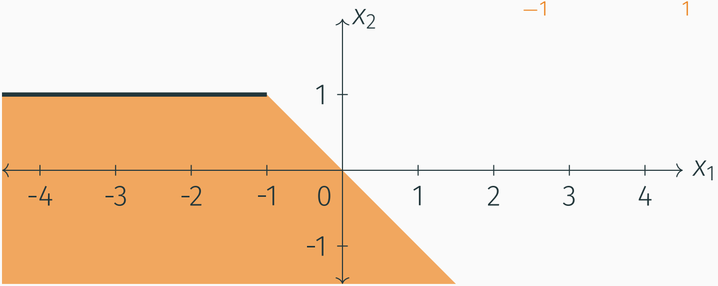




Interval Linear Programming: Example

Let's traverse through this!

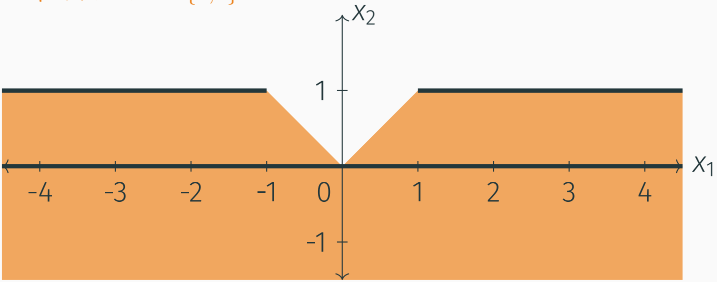
$$\begin{aligned} \text{maximize} \quad & x_2 \\ \text{subject to} \quad & [-1, 1]x_1 + x_2 \leq 0 \\ & x_2 \leq 1 \end{aligned}$$



Interval Linear Programming: Example

$$\begin{aligned} \text{maximize} \quad & x_2 \\ \text{subject to} \quad & [-1,1]x_1 + x_2 \leq 0 \\ & x_2 \leq 1 \end{aligned}$$

Optimal values: $\{0, 1\}$



Feasible solutions:

- Oettli–Prager: x solves $\boldsymbol{A}x = \boldsymbol{b} \Leftrightarrow |A_c x - b_c| \leq A_\Delta |x| + b_\Delta$
- Gerlach: x solves $\boldsymbol{A}x \leq \boldsymbol{b} \Leftrightarrow A_c x - A_\Delta |x| \leq \overline{b}$

Optimal values:

- Best optimal value (inequalities):
  For each $s \in \{\pm 1\}^n$ solve

$$\min\ (c_c - D_s c_\Delta)^T x \ \text{ s. t. } (A_c - A_\Delta D_s)x \leq \overline{b}, D_s x \geq 0$$

- Worst optimal value (inequalities):

$$\max\ \underline{b}^T y \ \text{ s. t. } \overline{A}^T y \leq \overline{c},\ \underline{A}^T y \geq \underline{c},\ y \leq 0$$

Optimal solutions:

- Special cases, approximations, ...

Dependency Problem (I)

$$\begin{aligned} \max \quad & x_1 \\ \text{s. t.} \quad & [0,1]x_1 - x_2 = 0, \\ & x_2 \leq 1, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Optimal set: $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in [1, \infty) \text{ and } x_2 = 1\}$

Dependency Problem (I)

$$\begin{aligned} \max \quad & x_1 \\ \text{s. t.} \quad & [0,1]x_1 - x_2 = 0, \\ & x_2 \leq 1, \\ & x_1, x_2 \geq 0. \end{aligned}$$

$$\begin{aligned} \max \quad & x_1 \\ \text{s. t.} \quad & [0,1]x_1 - x_2 \leq 0, \\ & [0,1]x_1 - x_2 \geq 0, \\ & x_2 \leq 1, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Optimal set: $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in [1, \infty) \text{ and } x_2 = 1\}$

Dependency Problem (I)

$$\begin{aligned} \max \quad & x_1 \\ \text{s. t.} \quad & [0,1]x_1 - x_2 = 0, \\ & x_2 \leq 1, \\ & x_1, x_2 \geq 0. \end{aligned}$$

$$\begin{aligned} \max \quad & x_1 \\ \text{s. t.} \quad & 1x_1 - x_2 \leq 0, \\ & 0x_1 - x_2 \geq 0, \\ & x_2 \leq 1, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Optimal set: $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in [1, \infty) \text{ and } x_2 = 1\}$

The solution $(0, 0)$ is now optimal, too!

Dependency Problem (I)

$$\begin{aligned} \max \quad & x_1 \\ \text{s. t.} \quad & [0,1]x_1 - x_2 = 0, \\ & x_2 \le 1, \\ & x_1, x_2 \ge 0. \end{aligned}$$

$$\begin{aligned} \max \quad & x_1 \\ \text{s. t.} \quad & [0,1]x_1 - x_2 \le 0, \\ & [0,1]x_1 - x_2 \ge 0, \\ & x_2 \le 1, \\ & x_1, x_2 \ge 0. \end{aligned}$$

Optimal set: $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in [1, \infty) \text{ and } x_2 = 1\}$

The solution $(0, 0)$ is now optimal, too!

...But the feasible set is the same! (Li, 2015)

Wei Li. A note on dependency between interval linear systems (2015).

Dependency Problem (II)

Example (Hladík, 2012)

$$[1,2]x \leq 2 \quad \rightarrow \quad [1,2]x^+ - [1,2]x^- \leq 2,\ x^+, x^- \geq 0$$

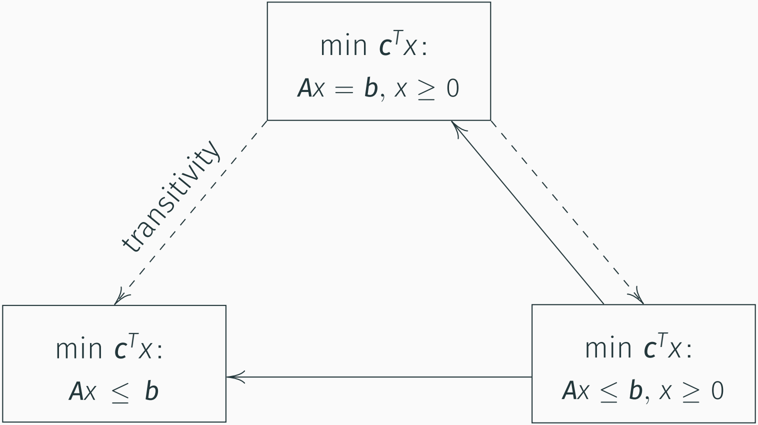
Original feasible set: $(-\infty, 2]$

Consider the new scenario $1x^+ - 2x^- \leq 2,\ x^+, x^- \geq 0$...

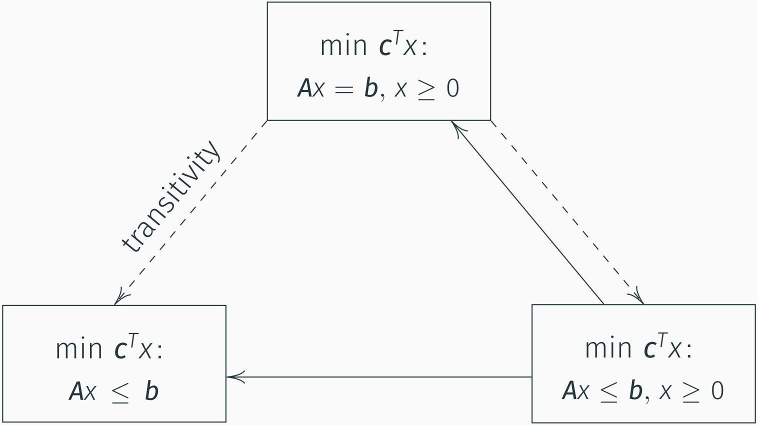
All real numbers are now feasible solutions, because we can express any real x as $x = x^+ - x^-$, where

$$x^+ = \max(2x, 0) \text{ and } x^- = |x|.$$

Transformations: The General Case



Transformations: The General Case



What if A is fixed?

Splitting Equations into Inequalities

Theorem 1

The optimal solution set of the interval linear program

$$\min \boldsymbol{c}^T x \colon Ax = \boldsymbol{b}, x \geq 0$$

is equal to the optimal solution set of the program

$$\min \boldsymbol{c}^T x \colon Ax \leq \boldsymbol{b}_1,\ -Ax \leq -\boldsymbol{b}_2,\ x \geq 0$$

with $\boldsymbol{b}_1 = \boldsymbol{b}_2 = \boldsymbol{b}$.

Splitting Equations into Inequalities

Theorem 1

The optimal solution set of the interval linear program

$$\min \boldsymbol{c}^T x \colon Ax = \boldsymbol{b}, x \geq 0$$

is equal to the optimal solution set of the program

$$\min \boldsymbol{c}^T x \colon Ax \leq \boldsymbol{b}_1,\ -Ax \leq -\boldsymbol{b}_2,\ x \geq 0$$

with $\boldsymbol{b}_1 = \boldsymbol{b}_2 = \boldsymbol{b}$.

Proof idea:

x^* optimal for scenario $\min c^T x \colon b_2 \leq Ax \leq b_1,\ x \geq 0$

$\Rightarrow Ax^* = b_3 \in \boldsymbol{b} \quad \Rightarrow x^*$ optimal for $\min c^T x \colon Ax = b_3,\ x \geq 0$

Imposing Non-negativity

Theorem 2

Let $\mathcal{S}$ denote the optimal solution set of $\min \boldsymbol{c}^T x\colon Ax \leq \boldsymbol{b}$ and let $\mathcal{S}'$ be the optimal solution set of the program

$$\min \boldsymbol{c}_1^T x^+ - \boldsymbol{c}_2^T x^- : Ax^+ - Ax^- \leq \boldsymbol{b},\ x^+, x^- \geq 0$$

with $\boldsymbol{c}_1 = \boldsymbol{c}_2 = \boldsymbol{c}$. Then, the following properties hold:

- If $x \in \mathcal{S}$, then there is $(x^+, x^-) \in \mathcal{S}'$ with $x = x^+ - x^-$.
- If $(x^+, x^-) \in \mathcal{S}'$, then $x^+ - x^- \in \mathcal{S}$.

Imposing Non-negativity

Theorem 2

Let $\mathcal{S}$ denote the optimal solution set of $\min \boldsymbol{c}^T x \colon Ax \leq \boldsymbol{b}$ and let $\mathcal{S}'$ be the optimal solution set of the program

$$\min \boldsymbol{c}_1^T x^+ - \boldsymbol{c}_2^T x^- : Ax^+ - Ax^- \leq \boldsymbol{b},\ x^+, x^- \geq 0$$

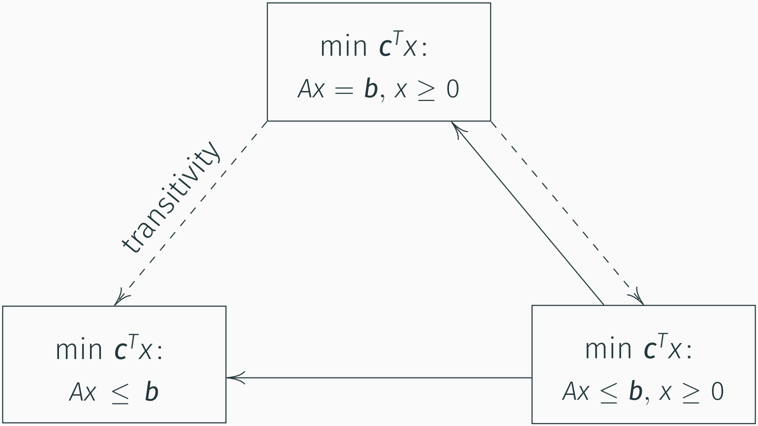
with $\boldsymbol{c}_1 = \boldsymbol{c}_2 = \boldsymbol{c}$. Then, the following properties hold:

- If $x \in \mathcal{S}$, then there is $(x^+, x^-) \in \mathcal{S}'$ with $x = x^+ - x^-$.
- If $(x^+, x^-) \in \mathcal{S}'$, then $x^+ - x^- \in \mathcal{S}$.

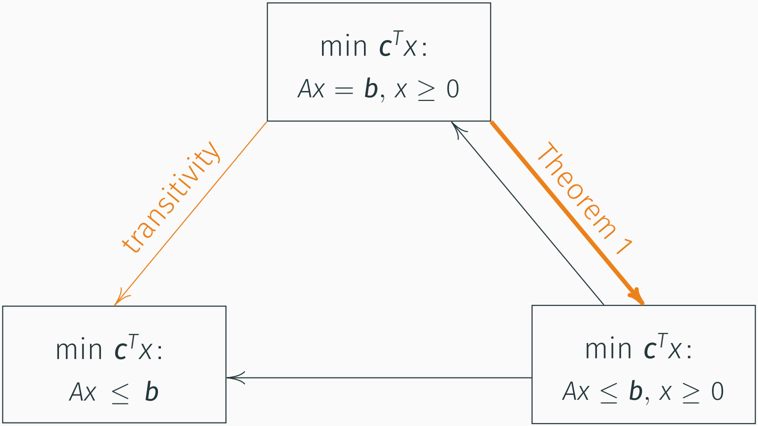
Proof idea:

For an optimal x^*, we have by dual feasibility some y^* with $A^T y^* = c_3 \in [c_2, c_1] \subseteq \boldsymbol{c}$. Then, x^* is optimal for $\min c_3^T x \colon Ax \leq b$.

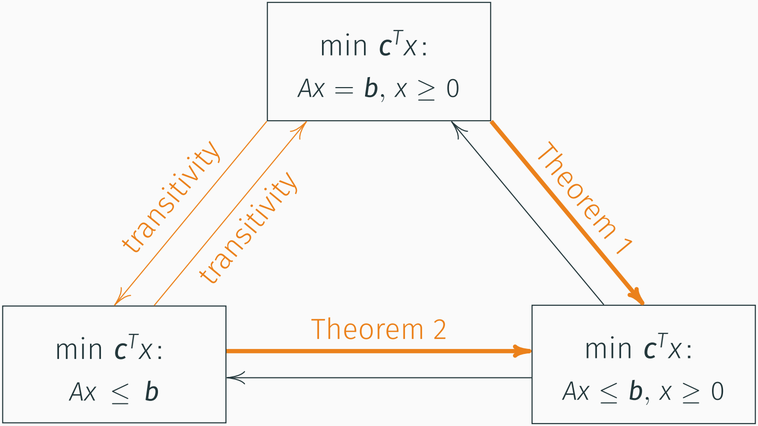
Transformations: The Special Case



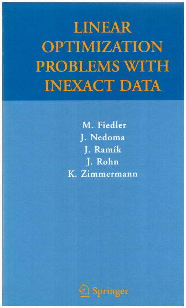
Transformations: The Special Case



Transformations: The Special Case

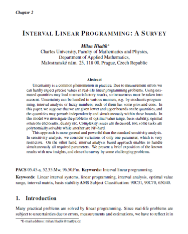


- In interval linear programming, the basic transformations may change the feasible/optimal set and other properties of a program.
- We have shown that the transformations do not affect the optimal set for problems with a fixed coefficient matrix (they may still change other properties!).
- Thus, we can directly generalize results concerning the optimal set of a particular type of programs to other types.



Linear Optimization Problems with Inexact Data (2006). Authors: M. Fiedler, J. Nedoma, J. Ramík, J. Rohn, K. Zimmermann

Interval linear programming: A survey (2012). M. Hladík. Linear Programming – New Frontiers in Theory and Applications.


Chapter 2

INTERVAL LINEAR PROGRAMMING: A SURVEY

*Milan Hladík**
Charles University, Faculty of Mathematics and Physics,
Department of Applied Mathematics,
Malostranské nám. 25, 118 00, Prague, Czech Republic

Abstract

Uncertainty is a common phenomenon in practice. Due to measurement errors we can hardly expect precise values in real-life linear programming problems. Using estimated quantities may lead to unsatisfactory results, so inexactness must be taken into account. Uncertainty can be handled in various manners, e.g. by stochastic programming, interval analysis or fuzzy numbers; each of them has some pros and cons. In this paper, we suppose that we are given lower and upper bounds on the quantities, and the quantities may perturb independently and simultaneously within these bounds. In this model we investigate the problems of optimal value range, basis stability, optimal solutions enclosures, duality etc. Complexity issues are discussed, too; some tasks are polynomially solvable while another are NP-hard.

This approach is more general and powerful than the standard sensitivity analysis. In sensitivity analysis, we consider variations of only one parameter, which is very restrictive. On the other hand, interval analysis based approach enables to handle simultaneously all required parameters. We present a brief exposition of the known results with new insights, and close the survey by some challenging problems.

PACS 05.45-a, 52.35.Mw, 96.50.Fm. **Keywords:** Interval linear programming.

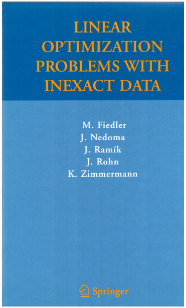
Keywords: Linear interval systems, linear programming, interval analysis, optimal value range, interval matrix, basis stability AMS Subject Classification: 90C31, 90C70, 65G40.

1. Introduction

Many practical problems are solved by linear programming. Since real-life problems are subject to uncertainties due to errors, measurements and estimations, we have to reflect it in

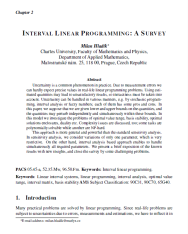
*E-mail address: milan.hladik@matfyz.cz

References



Linear Optimization Problems with Inexact Data (2006). Authors: M. Fiedler, J. Nedoma, J. Ramík, J. Rohn, K. Zimmermann

Interval linear programming: A survey (2012). M. Hladík. Linear Programming – New Frontiers in Theory and Applications.



Chapter 2

INTERVAL LINEAR PROGRAMMING: A SURVEY

*Milan Hladík**
Charles University, Faculty of Mathematics and Physics,
Department of Applied Mathematics,
Malostranské nám. 25, 118 00, Prague, Czech Republic

Abstract

Uncertainty is a common phenomenon in practice. Due to measurement errors we can hardly expect precise values in real-life linear programming problems. Using estimated quantities may lead to unsatisfactory results, so inexactness must be taken into account. Uncertainty can be handled in various manners, e.g. by stochastic programming, interval analysis or fuzzy numbers; each of them has some pros and cons. In this paper, we suppose that we are given lower and upper bounds on the quantities, and the quantities may perturb independently and simultaneously within these bounds. In this model we investigate the problems of optimal value range, basis stability, optimal solutions enclosures, duality etc. Complexity issues are discussed, too; some tasks are polynomially solvable while another are NP-hard.

This approach is more general and powerful than the standard sensitivity analysis. In sensitivity analysis, we consider variations of only one parameter, which is very restrictive. On the other hand, interval analysis based approach enables to handle simultaneously all required parameters. We present a brief exposition of the known results with new insights, and close the survey by some challenging problems.

PACS 05.45-a, 52.35.Mw, 96.50.Fm. **Keywords:** Interval linear programming.

Keywords: Linear interval systems, linear programming, interval analysis, optimal value range, interval matrix, basis stability AMS Subject Classification: 90C31, 90C70, 65G40.

1. Introduction

Many practical problems are solved by linear programming. Since real-life problems are subject to uncertainties due to errors, measurements and estimations, we have to reflect it in

*E-mail address: milan.hladik@matfyz.cz

Thank you for your attention!