(1) The art of formulating integer programs

Exercise 1.1. Charles wants to enroll in some of the classes $\{c_1, \ldots, c_5\}$. Using integer linear programming constraints, help him model the following requirements:

(a) he has to take at least two classes,

(b) if he takes c_1, then he also has to take c_5,

(c) if he takes c_2, then he cannot take c_4,

(d) he can take c_3 only if he also takes c_1 or c_2,

(e) he can take c_4 only if he also takes c_2 and c_3,

(f) if he takes two or more classes from the set $\{c_3, c_4, c_5\}$, then he cannot take c_2.

Exercise 1.2. Formulate the following constraints using integer linear programming:

(a) $x \in \{1, 2, 5, 22, 42\}$,

(b) $z = \min\{x, y\}$ for variables $x, y \in [-K, K]$,

(c) $y = |x|$ for variable $x \in [-K, K]$.

Exercise 1.3. Formulate an integer linear program for solving a given instance of the Sudoku puzzle.

Exercise 1.4. Formulate an integer linear program describing a union of k polytopes in the form

$$P^i = \left\{x \in \mathbb{R}^n : A^i x \leq b^i,\ 0 \leq x \leq u^i\right\}, \quad \text{for } i \in \{1, \ldots, k\}.$$

Exercise 1.5. Model the following graph problems as integer linear programs:

(a) minimum vertex cover (a set of vertices such that each edge is incident to some vertex in the set),

(b) maximum independent set (a subset of pairwise non-adjacent nodes),

(c) chromatic number (the smallest number of colors needed to color a graph).

Exercise 1.6. Model a given piecewise linear function $f(x)$ on an interval $[x_0, x_m]$ with breakpoints at $x_0, \ldots, x_m \in \mathbb{R}$ and values at the breakpoints $a_0, \ldots, a_m \in \mathbb{R}$.

Exercise 1.7. A company produces paper rolls in a uniform width of 100 cm and sells smaller rolls of widths 14 cm, 31 cm, 36 cm and 45 cm. Each 100 cm roll can be cut into two or more smaller rolls. The customer has ordered the following amounts of paper rolls:

Width	14	31	36	45
Quantity	211	395	610	97

Find a model minimizing the number of 100 cm rolls needed to satisfy the order and determine how to cut the rolls.