(2) Integer polyhedra, unimodularity & complexity

Exercise 2.1. Show that the following formulations are equivalent (they describe the same solution set):

- $P_1 = \{x \in \{0,1\}^4 : 12x_1 + 9x_2 + 6x_3 + 3x_4 \leq 14\}$,
- $P_2 = \{x \in \{0,1\}^4 : 4x_1 + 3x_2 + 2x_3 + x_4 \leq 4\}$,
- $P_3 = \{x \in \{0,1\}^4 : 4x_1 + 3x_2 + 2x_3 + x_4 \leq 4,\ x_1 + x_2 + x_3 \leq 1,\ x_1 + x_4 \leq 1\}$.

Which one would you choose for an integer linear programming model and why?

Exercise 2.2. Find the best formulation describing the set $\mathbb{Z}^2 \cap P$, where P is the polyhedron

$$\begin{pmatrix} -2 & -1 \\ 1 & -2 \\ -2 & 1 \\ 1 & 2 \end{pmatrix} x \leq \begin{pmatrix} -1 \\ 1 \\ 1 \\ 6 \end{pmatrix}.$$

Exercise 2.3. Are the following matrices totally unimodular?

$$\text{a)} \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix}, \quad \text{b)} \begin{pmatrix} 1 & 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 & 1 \\ 1 & 2 & 0 & -1 & -1 \\ 1 & 1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad \text{c)} \begin{pmatrix} 1 & 1 & 0 & -1 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Exercise 2.4. Let $A, B \in \mathbb{Z}^{n \times n}$ be unimodular matrices. Which of the matrices A^T, $A+B$, $A \cdot B$ are also unimodular? Do the same properties hold for total unimodularity?

Exercise 2.5. A transshipment problem is a modification of a transportation problem with additional nodes through which goods can be transshipped when they are transported from a supply point to a demand point.

(a) Formulate the transshipment problem as a network flow problem.

(b) Formulate the transshipment problem as a transportation problem.

Exercise 2.6. A matrix $A \in \mathbb{Z}^{m \times n}$ is said to have interval form, if all its rows are of the form $(0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)$, i.e. all 1-entries appear consecutively. Show that every interval-form matrix is also totally unimodular. *(Hint: Use elementary column operations.)*

Exercise 2.7. Let $\sigma(\cdot)$ denote the size of the binary representation of a given number. Show that

$$\sigma(a+b) \leq \sigma(a) + \sigma(b)$$

holds for each $a, b \in \mathbb{Z}$.