1 Interval arithmetic

Exercise 1. Prove the following properties of interval arithmetic for $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{IR}$:

a) $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{x} + \boldsymbol{z} \Rightarrow \boldsymbol{y} = \boldsymbol{z}$, [2 b]

b) $\boldsymbol{x} = x_c + x_\Delta[-1, 1]$. [2 b]

Exercise 2. The magnitude of an interval $\boldsymbol{x} \in \mathbb{IR}$ is defined as

$$\operatorname{mag}(\boldsymbol{x}) := \max\{|\overline{x}|, |\underline{x}|\}.$$

Show that we can also equivalently define the magnitude as $\operatorname{mag}(\boldsymbol{x}) = |x_c| + x_\Delta$. [2 b]

Exercise 3. Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathbb{IR}^{n\times n}$ be interval matrices.

a) Show that if $\boldsymbol{A} = A$ is a real matrix (with $A_\Delta = 0$), then $A(\boldsymbol{B} + \boldsymbol{C}) = A\boldsymbol{B} + A\boldsymbol{C}$. [2 b]

b) Decide whether the property $(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC})$ holds for interval matrix multiplication. [2 b]

c) Determine all interval matrices $\boldsymbol{A}$ satisfying $\boldsymbol{A} - \boldsymbol{A} = 0$ (and justify your answer). [2 b]

Exercise 4. Characterize the properties of algebraic structures $(\mathbb{IR}, +)$, $(\mathbb{IR}, -)$ and $(\mathbb{IR}, +, \cdot)$:

- Do the operations satisfy the associative, commutative and distributive property?
- Is there an identity (a neutral element) for the given operation?
- Do all intervals have an inverse with respect to the given operation? [6 b]

Exercise 5. Decide whether the interval function $f\colon \mathbb{IR} \to \mathbb{IR}$ defined as

$$f\left([a_c - a_\Delta, a_c + a_\Delta]\right) = \left[a_c - \frac{1}{2}a_\Delta, a_c + \frac{1}{2}a_\Delta\right]$$

is inclusion isotonic. [2 b]

Exercise 6. Let us define the distance between two intervals by the function

$$q(\boldsymbol{a}, \boldsymbol{b}) := \max(|\overline{a} - \overline{b}|, |\underline{a} - \underline{b}|).$$

Moreover, given two interval vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{IR}^n$, let the function $Q : \mathbb{IR}^n \times \mathbb{IR}^n \to \mathbb{R}$ be defined as

$$Q(\boldsymbol{x}, \boldsymbol{y}) = \max_{i=1,\dots,n} q(\boldsymbol{x}_i, \boldsymbol{y}_i).$$

Show that the function Q is a metric. [5 b]