4 Range of a function & nonlinear systems

You can utilize available tools for interval computations to aid in solving the numerical homework exercises in this set (e.g. interval libraries for the programming language of your choice, VERSOFT, LIME, INTLAB, ...). Own implementations of the methods will be rewarded with a higher point value. Please submit any scripts or pieces of source code used to solve the problems together with your solutions.

Exercise 1. Prove or disprove:

$$\text{a) } f(\cup_{i=1}^{n}\boldsymbol{x}_i) = \cup_{i=1}^{n} f(\boldsymbol{x}_i), \qquad \text{b) } f(\cap_{i=1}^{n}\boldsymbol{x}_i) = \cap_{i=1}^{n} f(\boldsymbol{x}_i). \qquad \text{[4 pts]}$$

Exercise 2. Consider a function with the following equivalent forms:

$$\begin{aligned} f(x) &= x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1, \\ g(x) &= 1 + x(-6 + x(15 + x(-20 + x(15 + x(-6 + x))))), \\ h(x) &= (x-1)^6. \end{aligned}$$

Compute interval enclosures $f(\boldsymbol{x})$, $g(\boldsymbol{x})$ and $h(\boldsymbol{x})$ for $\boldsymbol{x} = [0.999, 1.001]$. [2 pts]

Exercise 3. Derive the formulas to calculate slopes of the functions (with respect to $S_f(x, a)$):

(a) $f(x)^n$, [3 pts]

(b) $|f(x)|$. [3 pts]

Exercise 4. Compute an interval enclosure for the function

$$f(x) = \frac{x_1^3 - x_1 x_2}{x_1^2 + x_2^2}$$

over the interval box $\boldsymbol{x} = ([2-\delta, 2+\delta], [2-\delta, 2+\delta])$ for $\delta \in \{0.01, 0.1, 1\}$ using

(a) the natural interval extension,

(b) the mean value form,

(c) the slope form. [5 pts]

Exercise 5. Consider a wooden ball with a radius of 1 m and specific gravity of $g_{SP} = \frac{2}{3}$. (Specific gravity is the ratio of the density of the ball to the density of water $g_{SP} = \rho_{ball}/\rho_{water}$.)

The depth h to which the ball will sink in water is given by the equation

$$h^3 - 3h^2 + \frac{8}{3} = 0.$$

Find an interval enclosure for the value of h. [4 pts]

Exercise 6. Find an interval enclosure for the global minimization problem

$$\begin{aligned} &\min\ x_1 \\ &x_1^2 + x_2^2 \le 1, \\ &x_1^2 + x_2 = 0. \end{aligned} \qquad \text{[4 pts]}$$