Optimality and Boundedness in Interval Linear Programming

Complexity & Characterization

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Consider a linear programming problem...

$$\text{minimize } c^T x \text{ subject to } Ax \leq b$$

Interval Linear Programming

approximation and rounding

$b \approx 3.14159$

Consider a linear programming problem...

$$\text{minimize } c^T x \text{ subject to } Ax \leq b$$

estimating the future

€15.6 $\leq c \leq$ €17.1

representing missing data

$0.66,\ 0.21,\ 0.84,\ d = ?,\ 0.05$

discretization of time

$t_{min} = 22^\circ C,\ \ t_{max} = 23.5^\circ C$

inexact measurements

$a = 5 \pm 0.05g$

approximation and rounding

$[b] = [3.141592, 3.141593]$

Consider an interval linear programming problem...

$$\text{minimize } [c]^T x \text{ subject to } [A]x \leq [b]$$

estimating the future

$[c] = [15.6, 17.1]$

representing missing data

$[d] = [0, 1]$

inexact measurements

$[a] = [4.95, 5.05]$

discretization of time

$[t] = [22, 23.5]$

Interval Linear Programming: Definitions

- An interval linear program is a family of linear programs

$$\text{minimize } c^T x \text{ subject to } x \in \mathcal{M}(A, b),$$

  where $A \in [A], b \in [b], c \in [c]$ and $\mathcal{M}(A, b)$ is the feasible set.

- A linear program in the family is called a scenario.
- Usually, we consider one of the three main forms:
  1. minimize $[c]^T x$ subject to $[A]x = [b], x \geq 0,$
  2. minimize $[c]^T x$ subject to $[A]x \leq [b],$
  3. minimize $[c]^T x$ subject to $[A]x \leq [b], x \geq 0.$

Not equivalent!

Interval Linear Programming: Example

$$\begin{aligned} \text{maximize} \quad & x_2 \\ \text{subject to} \quad & [-1, 1]x_1 + x_2 \leq 0 \\ & x_2 \leq 1 \end{aligned}$$

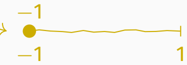
- What are the possible feasible/optimal solutions?
- What is the set of all optimal values?
- Are all scenarios of the interval program bounded?

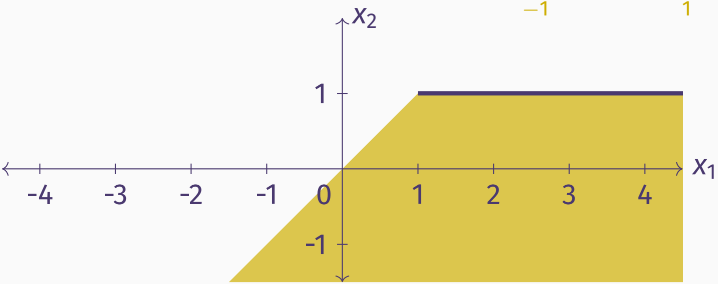
Vector x is a (weakly) feasible/optimal solution to an interval program, if x is a feasible/optimal solution for some scenario with $A \in [A]$, $b \in [b], c \in [c]$.

Interval Linear Programming: Example

$$\begin{aligned} \text{maximize} \quad & x_2 \\ \text{subject to} \quad & [-1, 1]x_1 + x_2 \leq 0 \\ & x_2 \leq 1 \end{aligned}$$

Let's traverse through this!





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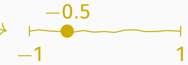
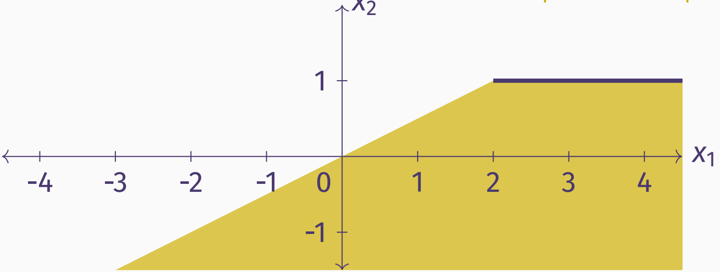




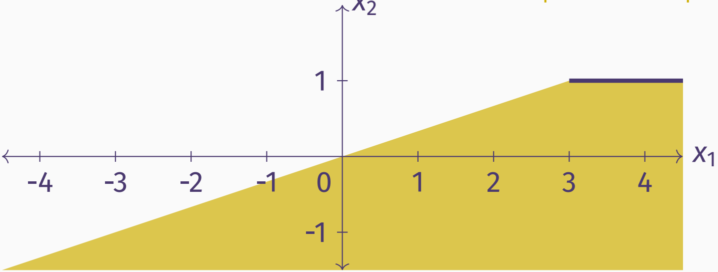
Interval Linear Programming: Example

Let's traverse through this!

$$\begin{aligned} \text{maximize} \quad & x_2 \\ \text{subject to} \quad & [-1, 1]x_1 + x_2 \leq 0 \\ & x_2 \leq 1 \end{aligned}$$

-0.33

-1 1



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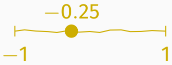

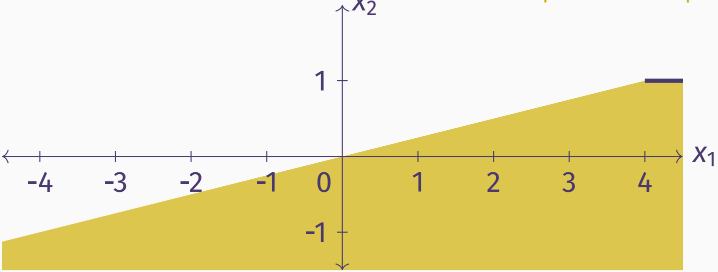

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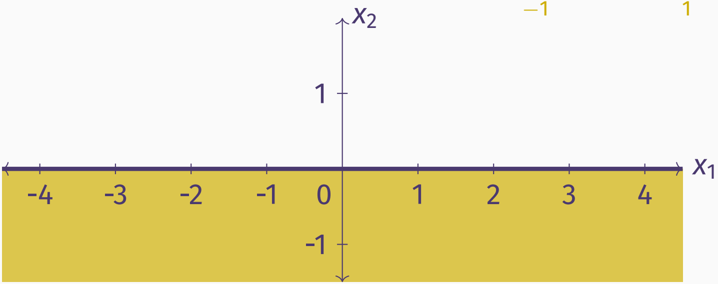


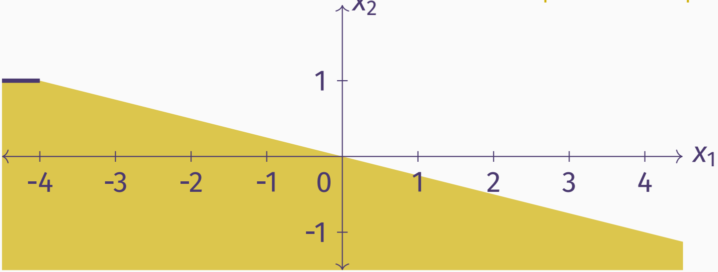
Interval Linear Programming: Example

Let's traverse through this!

$$\begin{aligned} \text{maximize} \quad & x_2 \\ \text{subject to} \quad & [-1, 1]x_1 + x_2 \le 0 \\ & x_2 \le 1 \end{aligned}$$

0.25

-1 1



Interval Linear Programming: Example

Let's traverse through this!

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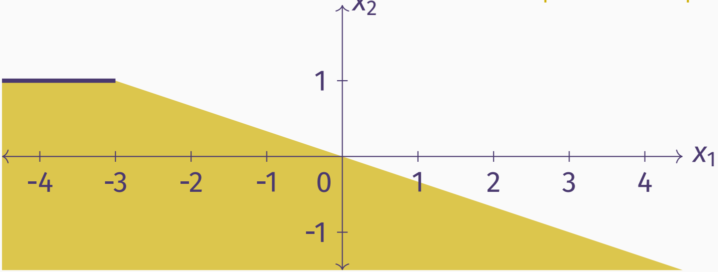

Interval Linear Programming: Example

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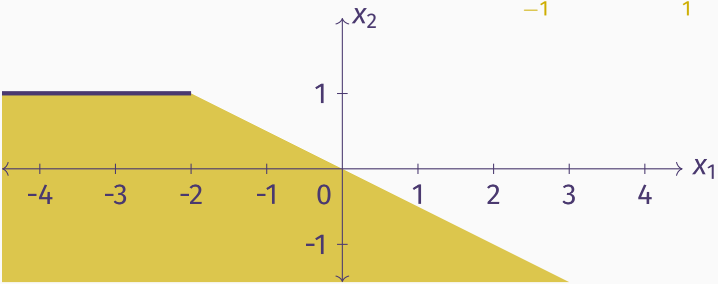

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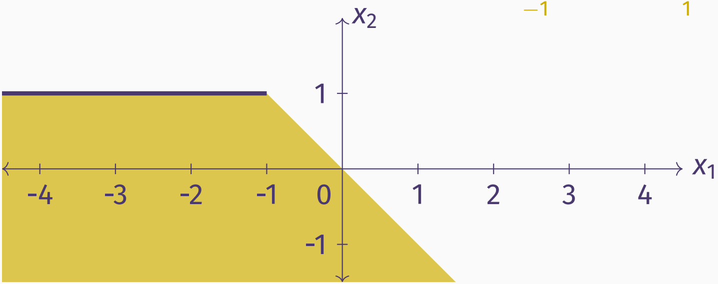
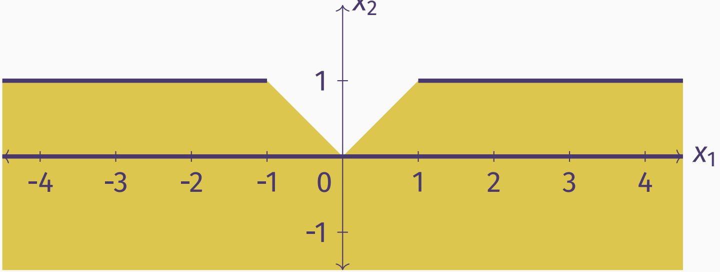


Interval Linear Programming: Example

$$\begin{aligned} \text{maximize} \quad & x_2 \\ \text{subject to} \quad & [-1,1]x_1 + x_2 \leq 0 \\ & x_2 \leq 1 \end{aligned}$$

Optimal values: $\{0, 1\}$



Dependency Problem

$$\begin{aligned} \max \quad & x_1 \\ \text{s. t.} \quad & [0,1]x_1 - x_2 = 0, \\ & x_2 \le 1, \\ & x_1, x_2 \ge 0. \end{aligned}$$

Optimal set: $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in [1, \infty) \text{ and } x_2 = 1\}$

Dependency Problem

$$\begin{aligned} \max \quad & x_1 \\ \text{s. t.} \quad & [0,1]x_1 - x_2 = 0, \\ & x_2 \leq 1, \\ & x_1, x_2 \geq 0. \end{aligned}$$

$$\begin{aligned} \max \quad & x_1 \\ \text{s. t.} \quad & [0,1]x_1 - x_2 \leq 0, \\ & [0,1]x_1 - x_2 \geq 0, \\ & x_2 \leq 1, \\ & x_1, x_2 \geq 0. \end{aligned}$$

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Dependency Problem

$$\begin{aligned} \max \quad & x_1 \\ \text{s. t.} \quad & [0,1]x_1 - x_2 = 0, \\ & x_2 \leq 1, \\ & x_1, x_2 \geq 0. \end{aligned} \qquad \begin{aligned} \max \quad & x_1 \\ \text{s. t.} \quad & 1x_1 - x_2 \leq 0, \\ & 0x_1 - x_2 \geq 0, \\ & x_2 \leq 1, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Optimal set: $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in [1, \infty) \text{ and } x_2 = 1\}$

The solution $(0, 0)$ is now optimal, too!

Weak/strong feasibility

- Is there a feasible scenario (a weakly feasible solution)?
- Is each scenario feasible?

Weak/strong unboundedness

- Is there a scenario with an unbounded objective value?
- Do all scenarios have an unbounded objective value?

Weak/strong optimality

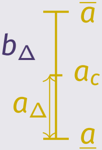
- Is there a scenario with an optimal solution?
- Do all scenarios have an optimal solution?

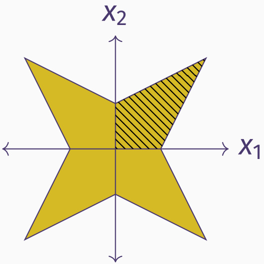
Weak Feasibility: Characterization

Theorem (Oettli–Prager, 1964; Gerlach, 1981)

$$x \in \mathbb{R}^n \text{ solves } [A]x = [b] \iff |A_c x - b_c| \leq A_\Delta |x| + b_\Delta$$

$$x \in \mathbb{R}^n \text{ solves } [A]x \leq [b] \iff A_c x - A_\Delta |x| \leq \overline{b}$$





For $x \geq 0$, we obtain a linear system!

Otherwise, we can use orthant decomposition.

Weak Feasibility: Complexity

Theorem (Gerlach, 1981)

$$x \in \mathbb{R}^n \text{ solves } [A]x \leq [b] \;\Leftrightarrow\; A_c x - A_\Delta |x| \leq \overline{b}$$

Theorem (Rohn, 2006)

Testing weak feasibility is NP-hard for interval linear systems of type $[A]x \leq [b]$.

Why?

Checking feasibility of a system of inequalities in the form

$$-e \leq Ax \leq e,\; e^T|x| \geq 1,$$

where $e = (1, \ldots, 1)^T$, is NP-hard. Apply Gerlach's theorem.

Strong Feasibility: Characterization

Theorem (Rohn, 1981)

An interval linear system in the form $[A]x = [b], x \geq 0$ is strongly feasible if and only if for each $p \in \{\pm 1\}^m$ the system

$$(A_c - diag(p)A_\Delta)x = b_c + diag(p)b_\Delta,\ x \geq 0$$

is feasible.

Theorem (Rohn & Kreslová, 1994)

An interval linear system in the form $[A]x \leq [b]$ is strongly feasible if and only if the system $\overline{A}x_1 - \underline{A}x_2 \leq \underline{b}, x_1 \geq 0, x_2 \geq 0$.

Strong Feasibility: Complexity

Theorem (Rohn, 2006)

Testing strong feasibility is co-NP-hard for interval linear systems of type $[A]x = [b], x \geq 0$.

Why?

$$[A]x = [b],\ x \geq 0 \text{ is weakly infeasible}$$
$$\Updownarrow$$
$$[A]^T y \geq 0,\ [b]^T y < 0 \text{ is weakly feasible}$$

Farkas' Lemma

Weak Unboundedness: Characterization

Theorem (Hladík, 2012)

An interval program in the form $\min [c]^T x : [A]x \leq [b], x \geq 0$ *is weakly unbounded if and only if the linear program* $\min \underline{c}^T x : \underline{A}x \leq \overline{b}, x \geq 0$ *is unbounded.*

Theorem

An interval program in the form $\min [c]^T x : [A]x \leq [b]$ *is weakly unbounded if and only if the interval linear program* $\min [c]^T x : [A]x \leq [b], diag(p)x \geq 0$ *is weakly unbounded for some* $p \in \{\pm 1\}^n$.

Weak Unboundedness: Complexity

Theorem

Testing weak unboundedness is NP-hard for interval linear programs of type $\min [c]^T x : [A]x \leq [b]$.

Why?

$$\min z : [A]x \leq [b] \text{ is weakly unbounded}$$

$$\Updownarrow$$

$$[A]x \leq [b] \text{ is weakly feasible}$$

Proved to be NP-hard

Weak Unboundedness: Complexity

Theorem

Testing weak unboundedness is NP-hard for interval linear programs of type $\min [c]^T x : [A]x \leq [b]$.

Why?

$$\min z : [A]x \leq [b] \text{ is weakly unbounded}$$
$$\Updownarrow$$
$$[A]x \leq [b] \text{ is weakly feasible}$$

Open problem: What about equations?

(Optimizing on the weakly feasible set is not sufficient.)

Strong Unboundedness: Characterization

Theorem (Hladík, 2012)

An interval linear program is strongly unbounded if and only if it is strongly feasible and its dual is not weakly feasible.

Theorem (Koníčková, 2006)

An interval linear program in the form

$$\min[c]^T x : [A]x = [b],\ x \geq 0$$

is strongly unbounded if and only if for each $p \in \{\pm 1\}^m$ *the linear program*

$$\min \underline{c}^T x : (A_c - diag(p)A_\Delta)x = b_c + diag(p)b_\Delta, x \geq 0$$

is unbounded.

Strong Unboundedness: Complexity

Theorem (Koníčková, 2006)

Testing strong unboundedness is co-NP-hard for interval linear programs of type $\min [c]^T x : [A]x = [b], x \geq 0$.

Why?

$$\max z : [A]x = [b],\ x \geq 0,\ z \geq 0 \text{ is strongly unbounded}$$

$$\Updownarrow$$

$$[A]x = [b],\ x \geq 0 \text{ is strongly feasible}$$

Proved to be co-NP-hard

Weak Optimality: Characterization

Lemma (Hladík, 2012)

An interval linear program is weakly optimal, if it is strongly feasible and its dual is weakly feasible, or vice versa.

Lemma (Hladík, 2012)

If an interval linear program is weakly optimal, then both the program itself and its dual are weakly feasible.

Weak feasibility of the interval linear program and its dual is not sufficient for weak optimality!

Weak Optimality: Complexity

Theorem

Testing weak optimality is NP-hard for all three basic types of interval linear programs.

Why?

1. $\min 0^T x : [A]x \leq [b]$ is weakly optimal
   $\Leftrightarrow [A]x \leq [b]$ is weakly feasible ← Proved to be NP-hard

Weak Optimality: Complexity

Theorem

Testing weak optimality is NP-hard for all three basic types of interval linear programs.

Why?

1. $\min 0^T x : [A]x \leq [b]$ is weakly optimal
   $\Leftrightarrow [A]x \leq [b]$ is weakly feasible

2. $\min [c]^T x : [A]x = [b], x \geq 0$ is weakly optimal
   $\Leftrightarrow \max [b]^T y : [A]^T y \leq [c]$ is weakly optimal

Weak Optimality: Complexity

Theorem

Testing weak optimality is NP-hard for all three basic types of interval linear programs.

Why?

1. $\min 0^T x : [A]x \leq [b]$ is weakly optimal
   $\Leftrightarrow [A]x \leq [b]$ is weakly feasible
2. $\min [c]^T x : [A]x = [b], x \geq 0$ is weakly optimal
   $\Leftrightarrow \max [b]^T y : [A]^T y \leq [c]$ is weakly optimal
3. We omit the proof for $\min [c]^T x : [A]x \leq [b], x \geq 0$.

Strong Optimality: Characterization

Theorem (Hladík, 2012)

An interval linear program is strongly optimal if and only if it is strongly feasible and its dual program also strongly feasible.

Therefore, we have...

$$\min [c]^T x : [A]x \leq [b],\ x \geq 0 \text{ is strongly optimal}$$
$$\Updownarrow$$
$$\overline{A}x \leq \underline{b},\ x \geq 0,\ \underline{A}^T y \leq \underline{c},\ y \leq 0 \text{ is feasible}$$

Strong Optimality: Complexity

Theorem

Testing strong optimality is co-NP-hard for interval programs of types $\min [c]^T x : [A]x = [b], x \geq 0$ and $\min [c]^T x : [A]x \leq [b]$.

Why?

1. $\min 0^T x : [A]x = [b],\ x \geq 0$ is strongly optimal
$\Leftrightarrow [A]x = [b],\ x \geq 0$ is strongly feasible

Proved to be co-NP-hard

Strong Optimality: Complexity

Theorem

Testing strong optimality is co-NP-hard for interval programs of types $\min [c]^T x : [A]x = [b], x \geq 0$ and $\min [c]^T x : [A]x \leq [b]$.

Why?

1. $\min 0^T x : [A]x = [b],\ x \geq 0$ is strongly optimal
$\Leftrightarrow [A]x = [b],\ x \geq 0$ is strongly feasible
2. $\min [c]^T x : [A]x \leq [b]$ is strongly optimal
$\Leftrightarrow \max [b]^T y : [A]^T y = [c], y \leq 0$ is strongly optimal

What about multiple criteria?

- H. Ishibuchi and H. Tanaka, Multiobjective programming in optimization of the interval objective function (1990).
- M. Hladík, Complexity of necessary efficiency in interval linear programming and multiobjective linear programming (2012).
- S. Rivaz and M. A. Yaghoobi, Weighted sum of maximum regrets in an interval MOLP problem (2015).
- C. O. Henriques and D. Coelho, A multiobjective interval portfolio model for supporting the selection of energy efficient lighting technologies (2017).
- C. O. Henriques and D. Coelho, Multiobjective Interval Transportation Problems: A Short Review (2017).
- ...

Conclusion

	$\min [c]^T x$ $[A]x = [b],\ x \geq 0$	$\min [c]^T x$ $[A]x \leq [b]$	$\min [c]^T x$ $[A]x \leq [b],\ x \geq 0$
strong feasibility	co-NP-hard	polynomial	polynomial
weak feasibility	polynomial	NP-hard	polynomial
strong unboundedness	co-NP-hard	polynomial	polynomial
weak unboundedness	?	NP-hard	polynomial
strong optimality	co-NP-hard	co-NP-hard	polynomial
weak optimality	NP-hard	NP-hard	NP-hard

Conclusion

	$\min [c]^T x$ $[A]x = [b],\ x \geq 0$	$\min [c]^T x$ $[A]x \leq [b]$	$\min [c]^T x$ $[A]x \leq [b],\ x \geq 0$
strong feasibility	co-NP-hard	polynomial	polynomial
weak feasibility	polynomial	NP-hard	polynomial
strong unboundedness	co-NP-hard	polynomial	polynomial
weak unboundedness	?	NP-hard	polynomial
strong optimality	co-NP-hard	co-NP-hard	polynomial
weak optimality	NP-hard	NP-hard	NP-hard

Thanks for your attention!