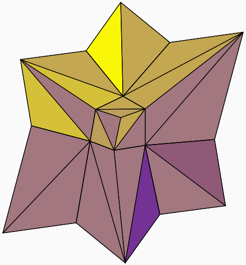

Towards the Extremes:

The Best Scenario of an Interval Linear Program

Elif Garajová, Milan Hladík

Department of Applied Mathematics
Faculty of Mathematics and Physics, Charles University, Prague

30th European Conference on Operational Research (EURO 2019)

Interval-valued uncertainty stems from rounding, estimations, measurement errors, discretization, etc.

Definition

Given two matrices $\underline{A}, \overline{A} \in \mathbb{R}^{m \times n}$ with $\underline{A} \leq \overline{A}$, we define an **interval matrix** $[A] = [\underline{A}, \overline{A}]$ as the set

$$\{A \in \mathbb{R}^{m \times n} : \underline{A} \leq A \leq \overline{A}\}.$$

For example $[A] = \begin{pmatrix} [0,1] & 2 \\ [3,5] & [-1,1] \end{pmatrix}$ or $[b] = \begin{pmatrix} [1,2] \\ [0,3] \\ [1,2] \end{pmatrix}$.

interval vector (interval box)

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Usually, we consider **weakly feasible solutions**, i.e. solutions that are feasible for at least one scenario $Ax \leq b$ of the interval system.

The weakly feasible set is not convex in general!

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Interval Linear System: An Example

For example, consider the interval linear system of equations:

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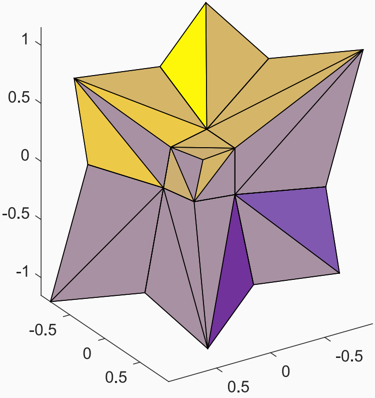

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Optimal Value Range

An interval linear program can have different "optimal" values

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we allow infinite values

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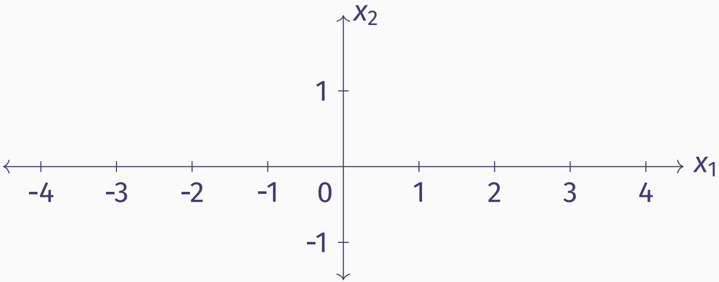

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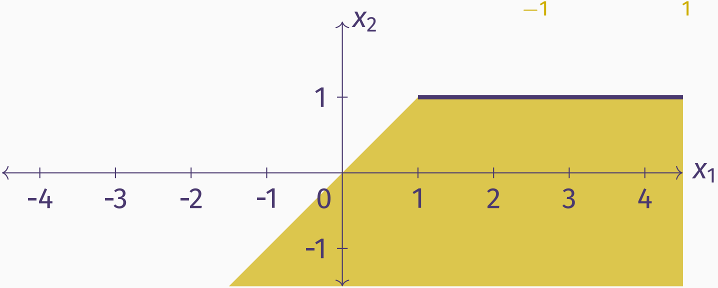

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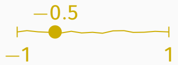

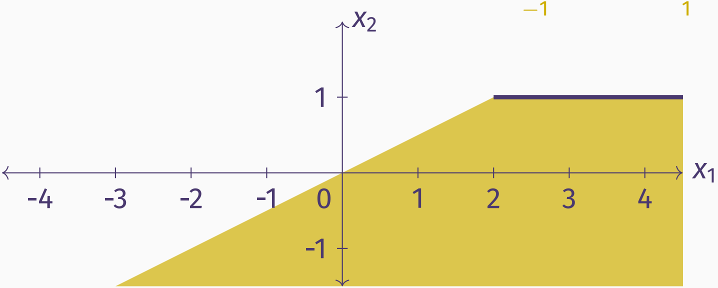

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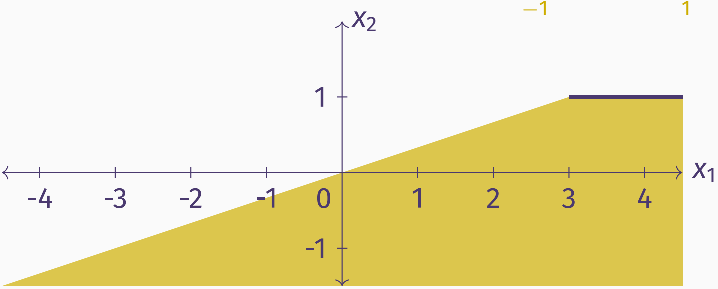

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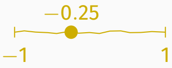

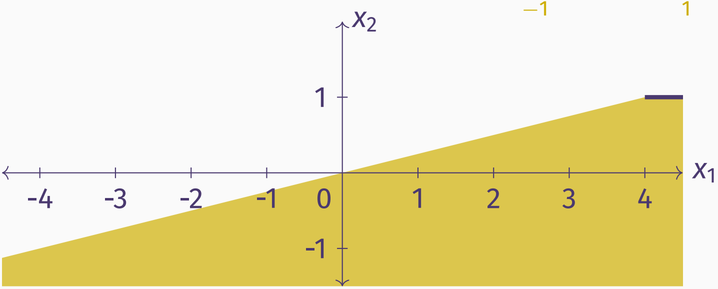

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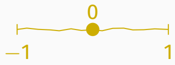

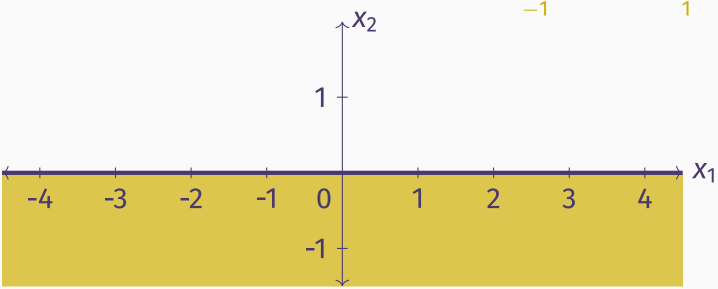

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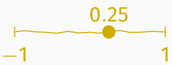

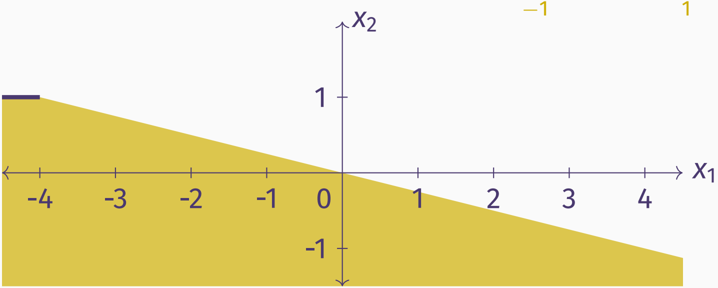

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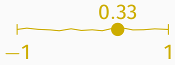

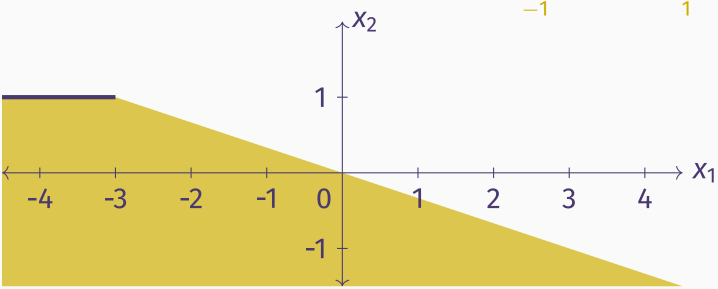

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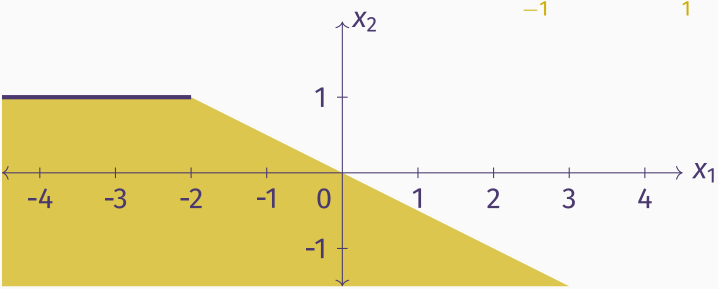

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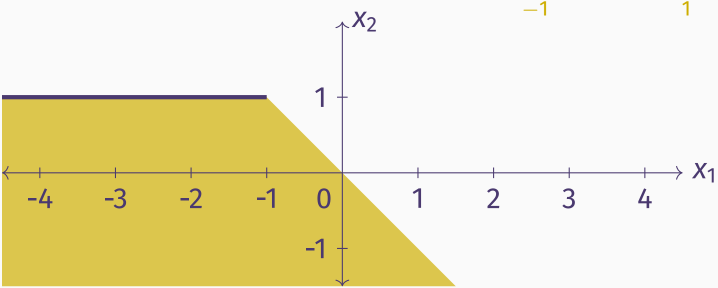

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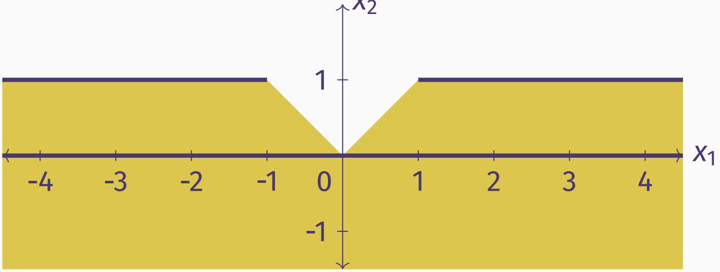

The Worst Case

Theorem (Rohn, 1984; Mráz, 1998)

The upper bound $\overline{f}$ on the optimal value range (i.e. the worst optimal value) can be computed as

$$\overline{f} = \sup_{s \in \{\pm 1\}^m} f(A_c - diag(s)A_\Delta, b_c + diag(s)b_\Delta, \overline{c}),$$

where $A_c = \frac{1}{2}(\overline{A} + \underline{A})$ and $A_\Delta = \frac{1}{2}(\overline{A} - \underline{A})$.

extremal scenarios

We also obtain the corresponding worst scenario:

$$A_c - \text{diag}(s)A_\Delta \in [A]$$
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The best optimal value $\underline{f}$ can be found by optimizing over the set of all weakly feasible solutions.

Theorem (Rohn, 1976)

The lower bound $\underline{f}$ on the optimal value range (i.e. the best optimal value) can be computed as

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This is not a scenario of $[A]x = [b], x \geq 0$.

So, what is the best scenario? And how "extremal" is it?

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Let $\underline{f}([A],[b],[c])$ *be finite and let* x^* *be an optimal solution of the problem* $\min \underline{c}^T x : \underline{A}x \leq \overline{b}, \overline{A}x \geq \underline{b}, x \geq 0$. *Then, we have*

$$\underline{f}([A],[b],[c]) = f(A_c - diag(s)A_\Delta, b_c + diag(s)b_\Delta, \underline{c}),$$

where

$$s_i = \begin{cases} \frac{(A_c x^* - b_c)_i}{(A_\Delta x^* + b_\Delta)_i}, & \text{if } (A_\Delta x^* + b_\Delta)_i > 0, \\ 1, & \text{if } (A_\Delta x^* + b_\Delta)_i = 0. \end{cases}$$

Since $s \notin \{\pm 1\}^m$, in general, this is not an extremal scenario.

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Theorem (Rohn, 2006)

Let the problem

$$\max \underline{b}^T p_1 - \overline{b}^T p_2 : \overline{A}^T p_1 - \underline{A}^T p_2 \leq \underline{c}, p_1 \geq 0, p_2 \geq 0$$

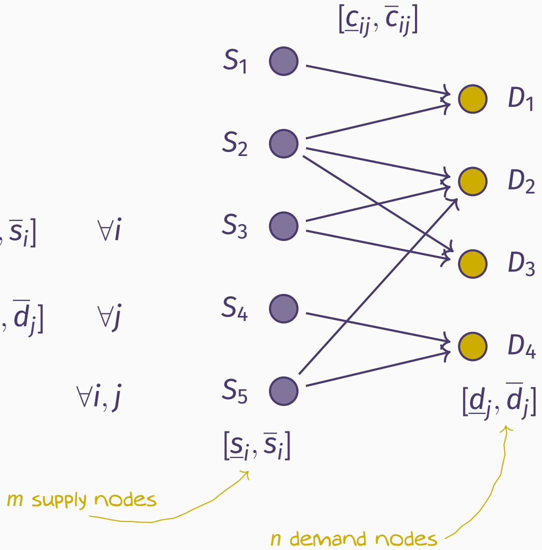
have an optimal solution p_1^*, p_2^* *satisfying* $p_1^* + p_2^* > 0$. *Then* $\underline{f}([A],[b],[c]) = f(A_c - diag(s)A_\Delta, b_c + diag(s)b_\Delta, \underline{c})$, *where*

$$s_i = \begin{cases} 1 & \text{if } (p_2^*)_i > 0, \\ -1 & \text{if } (p_2^*)_i = 0. \end{cases}$$

In this case, we have an extremal best scenario.

Interval Transportation Problem (ITP)

$$\begin{aligned} \text{minimize} \quad & \sum_{i,j} [\underline{c}_{ij}, \overline{c}_{ij}] x_{ij} \\ \text{subject to} \quad & \sum_{j} x_{ij} = [\underline{s}_i, \overline{s}_i] && \forall i \\ & \sum_{i} x_{ij} = [\underline{d}_j, \overline{d}_j] && \forall j \\ & x_{ij} \geq 0 && \forall i, j \end{aligned}$$



ITP: The Best Scenario

For the best scenario, we can fix $c = \underline{c}$, since we have $x \geq 0$.

We can also rewrite the interval problem as a linear program:

$$\sum_j x_{ij} = [\underline{s}_i, \overline{s}_i]$$

$$\begin{aligned}
\text{minimize} \quad & \sum_{i,j} \underline{c}_{ij} x_{ij} & \\
\text{subject to} \quad & \underline{s}_i \leq \sum_j x_{ij} \leq \overline{s}_i, & \forall i, \\
& \underline{d}_j \leq \sum_i x_{ij} \leq \overline{d}_j, & \forall j, \\
& x_{ij} \geq 0, & \forall i, j.
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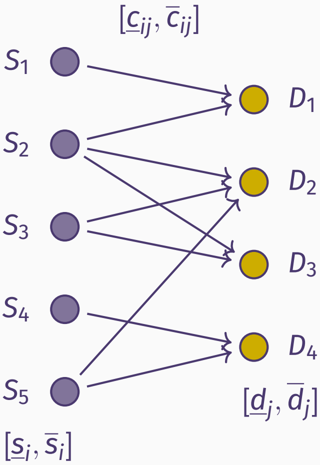
$$\sum_i x_{ij} = [\underline{d}_j, \overline{d}_j]$$

ITP: Towards the Extremes

Assume we have $\underline{s}, \underline{d} > 0$.



Then $x_{ij} > 0$ holds for at least $\max\{m, n\}$ variables x_{ij}, since we have to satisfy all suppliers and customers.

$\Rightarrow$ At most $2m + 2n - \max\{m, n\}$ slacks of the inequalities are non-zero in a (basic) solution, and at least $\max\{m, n\}$ slacks are zero.

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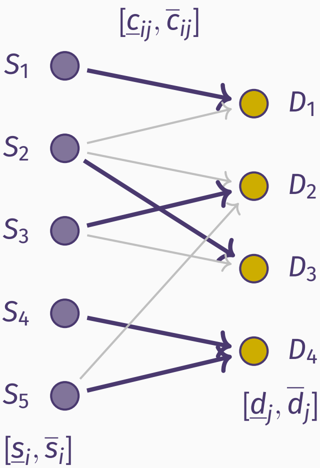

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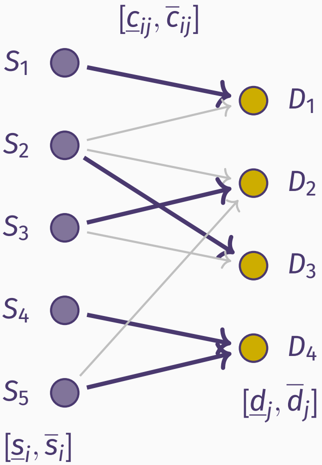

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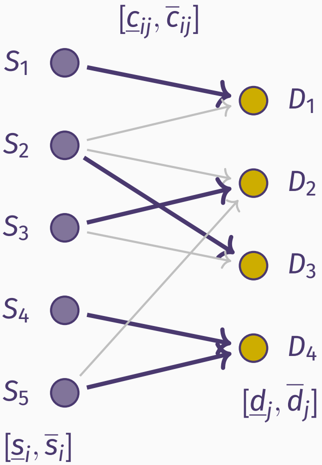

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- We considered the interval linear programming problem of computing the **optimal value range** and the corresponding **best and worst scenarios**.
- For interval programs in standard form, we can find an **extremal worst-case scenario**, in which all coefficients are set to their respective bounds.
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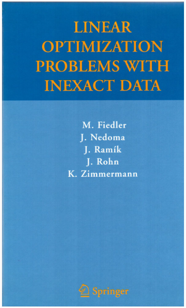
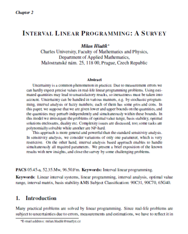


Linear Optimization Problems with Inexact Data (2006). Authors: M. Fiedler, J. Nedoma, J. Ramík, J. Rohn, K. Zimmermann



Chapter 2

INTERVAL LINEAR PROGRAMMING: A SURVEY

*Milan Hladík**
Charles University, Faculty of Mathematics and Physics,
Department of Applied Mathematics,
Malostranské nám. 25, 118 00, Prague, Czech Republic

Abstract

Uncertainty is a common phenomenon in practice. Due to measurement errors we can hardly expect precise values in real-life linear programming problems. Using estimated quantities may lead to unsatisfactory results, so inexactness must be taken into account. Uncertainty can be handled in various manners, e.g. by stochastic programming, interval analysis or fuzzy numbers; each of them has some pros and cons. In this paper, we suppose that we are given lower and upper bounds on the quantities, and the quantities may perturb independently and simultaneously within these bounds. In this model we investigate the problems of optimal value range, basis stability, optimal solutions enclosures, duality etc. Complexity issues are discussed, too; some tasks are polynomially solvable while another are NP-hard.

This approach is more general and powerful than the standard sensitivity analysis. In sensitivity analysis, we consider variations of only one parameter, which is very restrictive. On the other hand, interval analysis based approach enables to handle simultaneously all required parameters. We present a brief exposition of the known results with new insights, and close the survey by some challenging problems.

PACS 05.45-a, 52.35.Mw, 96.50.Fm. **Keywords:** Interval linear programming.

Keywords: Linear interval systems, linear programming, interval analysis, optimal value range, interval matrix, basis stability **AMS** Subject Classification: 90C31, 90C70, 65G40.

1. Introduction

Many practical problems are solved by linear programming. Since real-life problems are subject to uncertainties due to errors, measurements and estimations, we have to reflect it in

*E-mail address: milan.hladik@matfyz.cz

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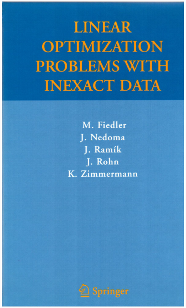
Slides will be available at http://elif.cz.

References



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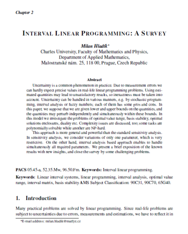

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