Solving Interval Linear Programs

From Theory to Algorithms

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Interval Uncertainty

Real-world optimization problems are often afflicted by uncertainty. Interval-valued uncertainty stems from:

- rounding errors,
- estimations and approximations,
- measurement errors,
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- ...

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Representing Interval Uncertainty

Definition

Given two matrices $\underline{A}, \overline{A} \in \mathbb{R}^{m \times n}$ with $\underline{A} \leq \overline{A}$, we define an **interval matrix** $[A] = [\underline{A}, \overline{A}]$ as the set

$$\{A \in \mathbb{R}^{m \times n} : \underline{A} \leq A \leq \overline{A}\}.$$

Analogously, we define an **interval vector (box)** $[b] = [\underline{b}, \overline{b}]$ for $\underline{b}, \overline{b} \in \mathbb{R}^n$ with $\underline{b} \leq \overline{b}$ as the set

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For example $[A] = \begin{pmatrix} [0,1] & 2 \\ [3,5] & [-1,1] \end{pmatrix}$ or $[b] = \begin{pmatrix} [1,2] \\ [0,3] \\ [1,2] \end{pmatrix}$.

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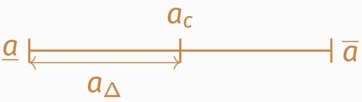

center $A_c = \frac{1}{2}(\overline{A} + \underline{A})$

radius $A_\Delta = \frac{1}{2}(\overline{A} - \underline{A})$

Interval Linear Programming

Definition

Given $[A] \in \mathbb{IR}^{m \times n}, [b] \in \mathbb{IR}^m, [c] \in \mathbb{IR}^n$, we define an **interval linear program** (in the standard form)

$$\text{minimize } [c]^T x \text{ subject to } [A]x = [b], x \geq 0$$

as the set of all linear programs in the form minimize $c^T x$ subject to $Ax = b, x \geq 0$ with $A \in [A], b \in [b], c \in [c]$.

A solution $x \in \mathbb{R}^n$ is called **weakly feasible/optimal**, if it is feasible/optimal for some scenario of the interval linear program.

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Let us consider a linear program with an interval equation...

$$
\begin{aligned}
\max \quad & x_1 \\
\text{s. t.} \quad & [0, 1]x_1 - x_2 = 0, \\
& x_2 \leq 1, \\
& x_1, x_2 \geq 0.
\end{aligned}
$$

Optimal set: $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in [1, \infty)$ and $x_2 = 1\}$

Dependency Problem

...and split the equation into two inequalities.

$$\begin{aligned} \max \quad & x_1 \\ \text{s. t.} \quad & [0,1]x_1 - x_2 = 0, \\ & x_2 \leq 1, \\ & x_1, x_2 \geq 0. \end{aligned}$$

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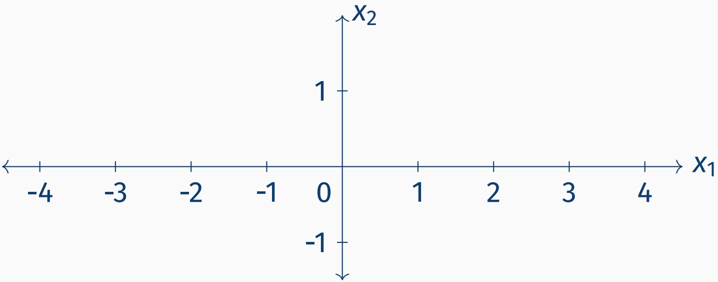
$$\begin{aligned} \max \quad & x_1 \\ \text{s. t.} \quad & 1x_1 - x_2 \leq 0, \\ & 0x_1 - x_2 \geq 0, \\ & x_2 \leq 1, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Optimal set: $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in [1, \infty) \text{ and } x_2 = 1\}$

The solution $(0, 0)$ is now optimal, too!

Interval Linear Program: An Example

$$\begin{aligned} \text{maximize} \quad & x_2 \\ \text{subject to} \quad & [-1, 1]x_1 + x_2 \le 0 \\ & x_2 \le 1 \end{aligned}$$



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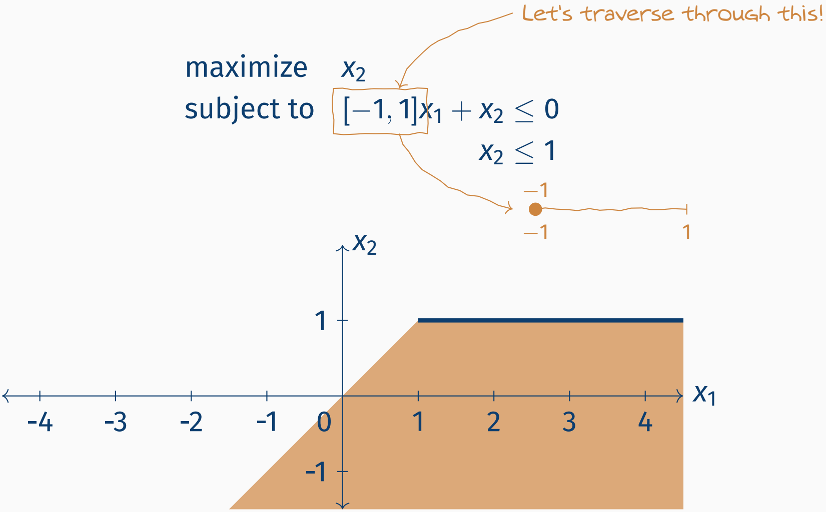

Interval Linear Program: An Example

Let's traverse through this!

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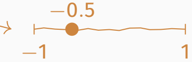

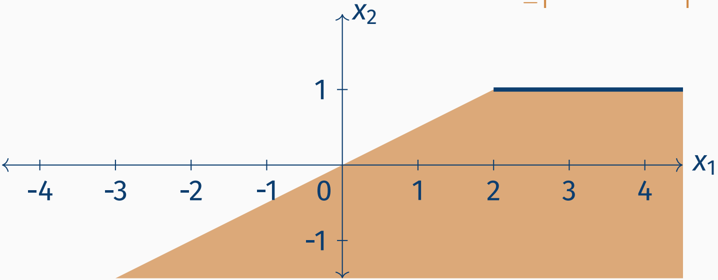

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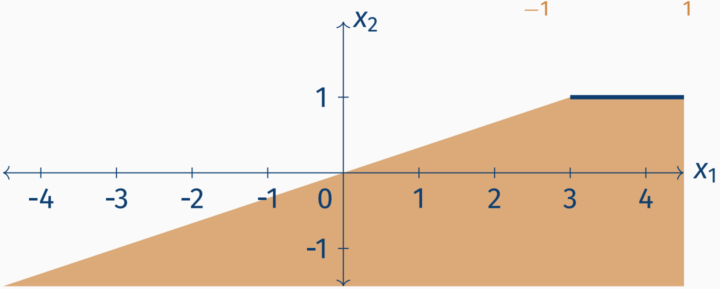

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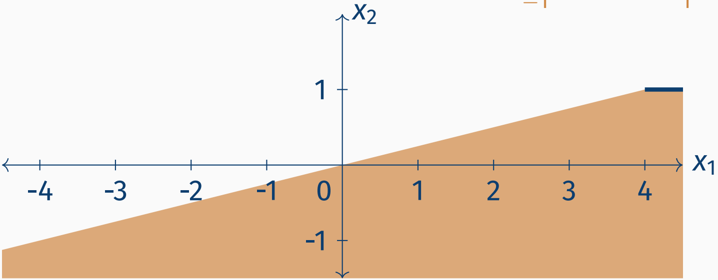

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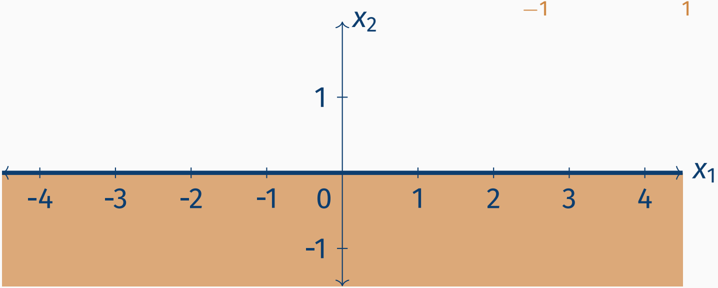

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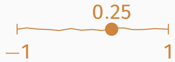

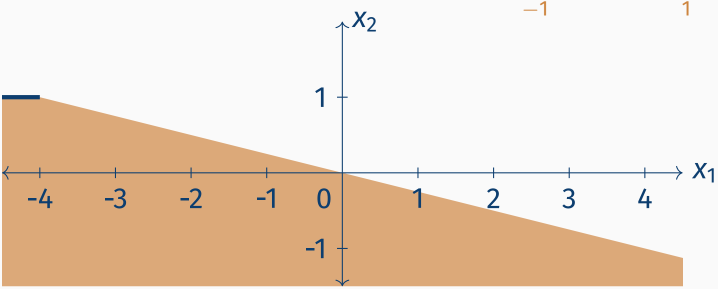

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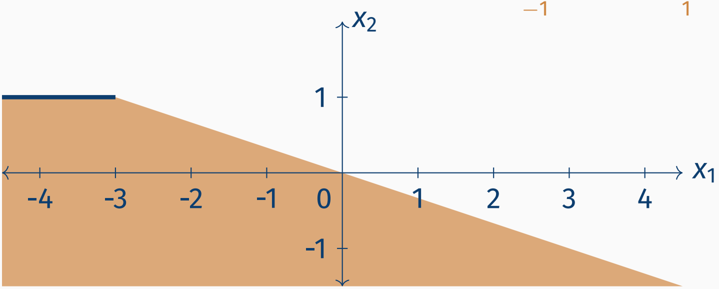

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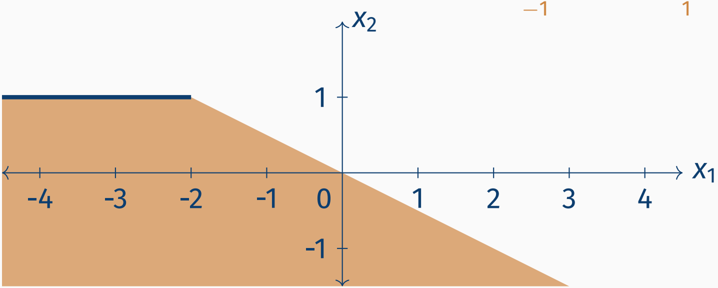

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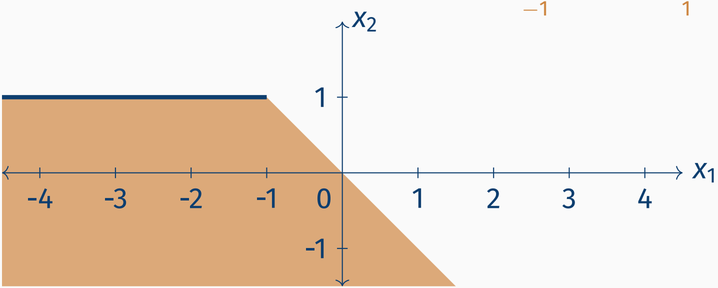

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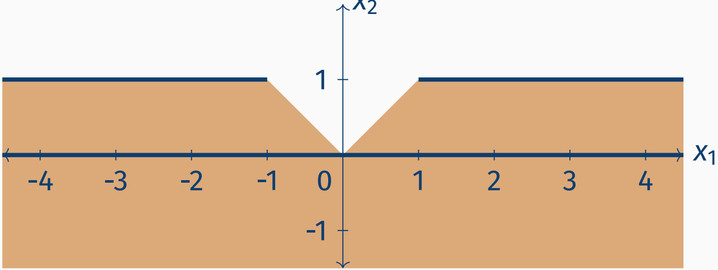
Optimal values: $\{0, 1\}$

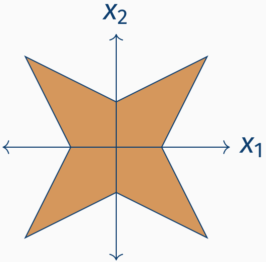


Weakly Feasible Solutions

Oettli–Prager (1964), Gerlach (1981)

$$x \in \mathbb{R}^n \text{ solves } [A]x = [b] \iff |A_c x - b_c| \leq A_\Delta |x| + b_\Delta$$

$$x \in \mathbb{R}^n \text{ solves } [A]x \leq [b] \iff A_c x - A_\Delta |x| \leq \overline{b}$$



The feasible solution set is not convex, in general.

But, it becomes convex when restricted to an orthant.

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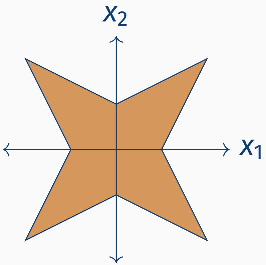

Orthant decomposition (ineq.):

Given a signature $s \in \{\pm 1\}^n$, the corresponding orthant is the set

$$\{x \in \mathbb{R}^n : \operatorname{diag}(s)x \ge 0\}.$$

Furthermore, we have $|x| = \operatorname{diag}(s)x$.

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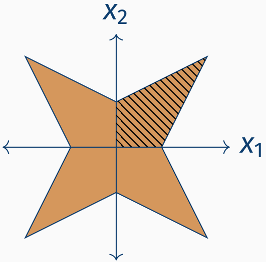

Orthant decomposition (ineq.):

For $x \geq 0$ we have the feasible set

$$A_c x - A_\Delta x \leq \overline{b},$$

or $\underline{A} x \leq \overline{b}$.

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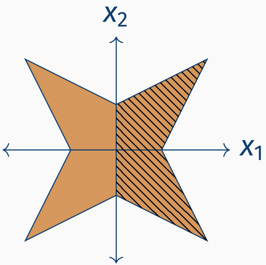

Orthant decomposition (ineq.):

In general, for $\operatorname{diag}(s)x \geq 0$ we have the feasible set

$$A_c x - A_\Delta \operatorname{diag}(s) x \leq \overline{b}.$$

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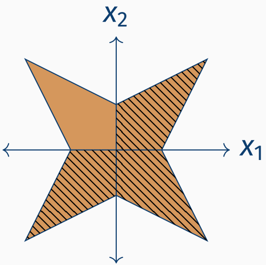

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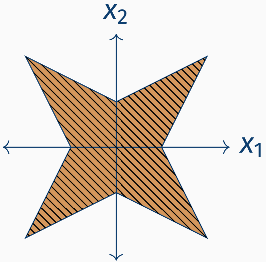

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Weakly Optimal Solutions

Using duality of classical linear programming, we can obtain a parametric characterization of the optimal solution set

$$
\begin{aligned}
&Ax = b,\ x \geq 0,\\
&A^T y \leq c,\\
&c^T x = b^T y,\\
&A \in [A], b \in [b], c \in [c].
\end{aligned}
$$

Not an interval linear program!

In general, the optimal solution set may be complicated (non-convex, disconnected). However, for some special cases, we can derive stronger characterizations.

Special Case: Fixed Matrix

For a fixed constraint matrix, we can describe the weakly optimal solution set by the non-linear system

$$Ax = b,\ x \geq 0, A^T y \leq c, x^T(c - A^T y) = 0,$$
$$\underline{b} \leq b \leq \overline{b},\ \underline{c} \leq c \leq \overline{c}.$$

Now, complementary slackness can be equivalently restated as $\forall i \in \{1, \ldots, n\} : x_i = 0 \vee z_i = 0$ with $z = c - A^T y$.

Theorem

The set of weakly optimal solutions of the interval LP

$$\begin{aligned} \text{minimize} \quad & [c]^T x \\ \text{subject to} \quad & Ax = [b], x \geq 0 \end{aligned}$$

is a union of at most 2^n convex polyhedra.

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Special Case: Basis Stability

Definition

Given a basis $B \subseteq \{1, \ldots, n\}$, an interval linear program

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is **B-stable**, if B is an optimal basis for each scenario.

Theorem (Beeck, 1978; Hladík, 2014)

Under unique B-stability, the set of all weakly optimal solutions is

$$\underline{A}_B x_B \leq \overline{b},\ -\overline{A}_B x_B \leq -\underline{b},\ x_B \geq 0,\ x_N = 0.$$

But, B-stability is NP-hard to test!

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By convention, we say that a problem “maximize $f(x)$ subject to $x \in X$” is NP-hard, if the corresponding decision problem “Is $f(x) \geq r$ for some $x \in X$?” is NP-hard.

Theorem

Let $\mathcal{S}(A, [b], c)$ denote the optimal set of an interval LP

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = [b], x \geq 0. \end{aligned}$$

Then, the problem

$$\begin{aligned} \text{optimize} \quad & x_i \\ \text{subject to} \quad & x \in \mathcal{S}(A, [b], c) \end{aligned}$$

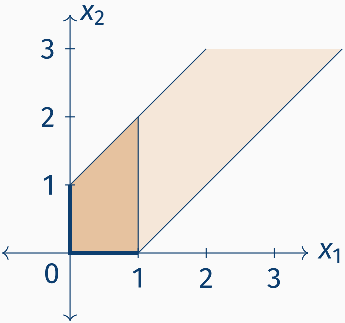
for $i \in \{1, \ldots, n\}$ is NP-hard.

To obtain a simpler approximation of the optimal set, we can relax the dependencies in the parametric description and consider the corresponding interval linear program

$$[A]x = [b],\ x \geq 0, [A]^T y \leq [c], [c]^T x = [b]^T y.$$

Example:

$$\begin{aligned} \text{minimize} \quad & x_1 \\ \text{subject to} \quad & x_1 - x_2 = [-1, 1], \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned}$$



Orthant decomposition:

For signatures $s \in \{\pm 1\}^m$ solve

$c_c^T x - b_c^T y \le c_\Delta^T x + b_\Delta^T \operatorname{diag}(s) y,\ c_c^T x - b_c^T y \ge -c_\Delta^T x - b_\Delta^T \operatorname{diag}(s) y,$

$\underline{A} x \le \overline{b},\ -\overline{A} x \le -\underline{b},\ x \ge 0,$

$A_c^T y - A_\Delta^T \operatorname{diag}(s) y \le \overline{c}, \operatorname{diag}(s) y \ge 0.$

Decomposition by complementarity:

For an index set $I \subseteq \{1, \dots, n\}$ solve

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&[A]x = [b], \\
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Orthant decomposition: $[c]^T x = [b]^T y$

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Linearization (Convexification) Methods

We have the interval relaxation to describe the optimal set:

$$[A]x = [b],\ x \geq 0, [A]^T y \leq [c], [c]^T x = [b]^T y.$$

Combining it with the Oettli–Prager and Gerlach theorems, we obtain a system with absolute-value non-linearities.

Theorem (Beaumont, 1998; Hladík, 2012)

Let $\mathbf{y} = [\underline{y}, \overline{y}] \in \mathbb{IR}$ *with* $\underline{y} < \overline{y}$. *Then for every* $y \in \mathbf{y}$ *it holds that*

$$|y| \leq \alpha y + \beta,$$

where

$$\alpha = \frac{|\overline{y}| - |\underline{y}|}{\overline{y} - \underline{y}}, \quad \beta = \frac{\overline{y}|\underline{y}| - \underline{y}|\overline{y}|}{\overline{y} - \underline{y}}.$$

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Branch-and-Bound Methods

Approximating the optimal set by an **interval box** (or a **convex polyhedron**) can lead to significant overestimation.

To obtain a tighter approximation, we can also describe the (non-convex) set by a **subpaving**, i.e. a **union of interval boxes**.

Branch-and-bound interval methods have been successfully applied in solving non-linear constraints and linear parametric systems yielding a subpaving for the described feasible set.

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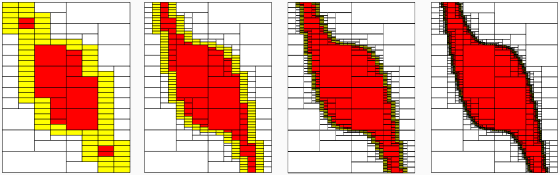

Conclusion

- We consider the problem of characterizing the set of all **weakly optimal solutions** of an **interval linear program**.
- Several methods for approximating the optimal set have been proposed throughout the years, such as enclosures of the **interval relaxation, orhant** or **complementarity decomposition** or iterative **linearization-based** algorithms.
- As using interval boxes or general convex polyhedra may lead to high overestimation of the set, applying a **branch-and-bound** method to describe the set by a **union of boxes** may be beneficial.

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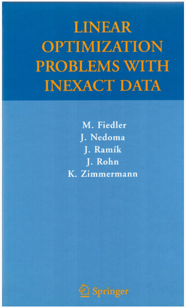

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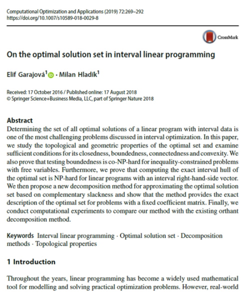

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On the optimal solution set in interval linear programming

Elif Garajová[1] · Milan Hladík[1]

Received: 17 October 2016 / Published online: 17 August 2018

Abstract
Determining the set of all optimal solutions of a linear program with interval data is one of the most challenging problems discussed in interval optimization. In this paper, we study the topological and geometric properties of the optimal set and examine sufficient conditions for its closedness, boundedness, connectedness and convexity. We also prove that testing boundedness is co-NP-hard for inequality-constrained problems with free variables. Furthermore, we prove that computing the exact interval hull of the optimal set is NP-hard for linear programs with an interval right-hand-side vector. We then propose a new decomposition method for approximating the optimal solution set based on complementary slackness and show that the method provides the exact description of the optimal set for problems with a fixed coefficient matrix. Finally, we conduct computational experiments to compare our method with the existing orthant decomposition method.

Keywords Interval linear programming · Optimal solution set · Decomposition methods · Topological properties

1 Introduction

Throughout the years, linear programming has become a widely used mathematical tool for modelling and solving practical optimization problems. However, real-world

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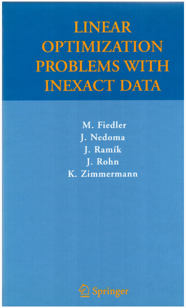

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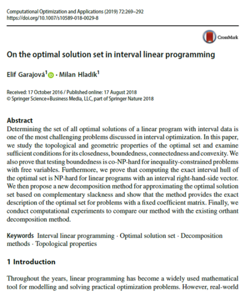

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