Approximating the Optimal Value Range in Interval Linear Programming

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Consider a linear programming problem...

$$\text{minimize } c^T x \text{ subject to } Ax \leq b$$

approximation and rounding

$b \approx 3.14159$

Consider a linear programming problem...

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estimating the future

€$25.6 \leq c \leq$ €27.1

inexact measurements

$a = 5 \pm 0.05g$

Interval Linear Programming

approximation and rounding

$\boldsymbol{b} = [3.141592, 3.141593]$

Consider an *interval* linear programming problem...

$$\text{minimize } \boldsymbol{c}^T x \text{ subject to } \boldsymbol{A}x \leq \boldsymbol{b}$$

estimating the future

$\boldsymbol{c} = [25.6, 27.1]$

inexact measurements

$\boldsymbol{a} = [4.95, 5.05]$

Interval Linear Programming: Example

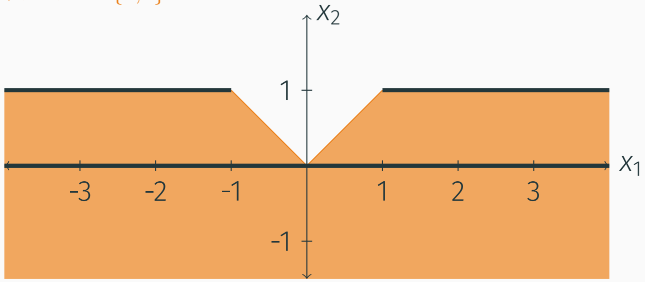
$$\begin{aligned} \text{maximize} \quad & x_2 \\ \text{subject to} \quad & [-1, 1]x_1 + x_2 \leq 0 \\ & x_2 \leq 1 \end{aligned}$$

- What are the possible feasible solutions?
- Which solutions are optimal for some scenario?
- What is the set/range of all optimal values?

Interval Linear Programming: Example

$$\begin{aligned} \text{maximize} \quad & x_2 \\ \text{subject to} \quad & [-1, 1]x_1 + x_2 \leq 0 \\ & x_2 \leq 1 \end{aligned}$$

Optimal values: $\{0, 1\}$



Optimal Value Range

Optimal value of an LP: $f(A, b, c) = \inf\{c^T x : Ax \leq b\}$

- $f(A, b, c) = -\infty$ if it is unbounded,
- $f(A, b, c) = \infty$ if it is infeasible,
- $f(A, b, c) = c^T x^*$ if there is an optimal solution x^*.

Lower bound of the optimal value range:

$$\underline{f}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}) = \inf\{f(A, b, c) : A \in \boldsymbol{A}, b \in \boldsymbol{b}, c \in \boldsymbol{c}\}$$

Upper bound of the optimal value range:

$$\overline{f}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}) = \sup\{f(A, b, c) : A \in \boldsymbol{A}, b \in \boldsymbol{b}, c \in \boldsymbol{c}\}$$

Computational Complexity

	$\min \boldsymbol{c}^T x$ $Ax = \boldsymbol{b},\ x \geq 0$	$\min \boldsymbol{c}^T x$ $Ax \leq \boldsymbol{b}$	$\min \boldsymbol{c}^T x$ $Ax \leq \boldsymbol{b},\ x \geq 0$
Best opt. value $\underline{f}$	polynomial	NP-hard	polynomial
Worst opt. value $\overline{f}$	NP-hard	polynomial	polynomial

Dependency problem:

$$\max x \text{ subject to } x = [0,1] \ \rightarrow\ \max x \text{ subject to } x \leq [0,1],\ x \geq [0,1]$$

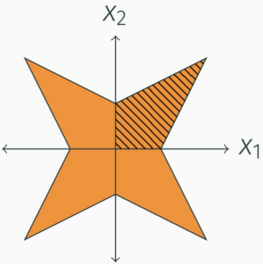
The optimal value range changes from $[0,1]$ to $[0,\infty)$, because the multiple occurrences of a coefficient are independent!

Describing the Feasible Set

Oettli–Prager (1964), Gerlach (1981)

$$x \in \mathbb{R}^n \text{ solves } \boldsymbol{A}x = \boldsymbol{b} \;\Leftrightarrow\; |A_c x - b_c| \leq A_\Delta |x| + b_\Delta$$

$$x \in \mathbb{R}^n \text{ solves } \boldsymbol{A}x \leq \boldsymbol{b} \;\Leftrightarrow\; A_c x - A_\Delta |x| \leq \overline{b}$$



The Polynomial Problems

- Best optimal value $\underline{f}$ of min $\boldsymbol{c}^T x : \boldsymbol{A}x \leq \boldsymbol{b},\, x \geq 0$

  Non-negative variables $\Rightarrow$ objective vector $\underline{c}$

  Largest feasible set: $\underline{A}x \leq \overline{b},\, x \geq 0$

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  Oettli–Prager: $\underline{A}x \leq \overline{b},\ \overline{A}x \geq \underline{b},\ x \geq 0$

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  Oettli–Prager: $\underline{A}x \leq \overline{b},\ \overline{A}x \geq \underline{b},\ x \geq 0$

- Worst optimal value $\overline{f}$ of min $\boldsymbol{c}^T x : \boldsymbol{A}x \leq \boldsymbol{b}$

  $\rightarrow$ Dual program: max $\boldsymbol{b}^T y : \boldsymbol{A}^T y = \boldsymbol{c},\ y \leq 0$

The Hard Problems

- Worst optimal value $\overline{f}$ of min $\boldsymbol{c}^T x : \boldsymbol{A}x = \boldsymbol{b},\ x \geq 0$

  $\rightarrow$ by duality

- Best optimal value $\underline{f}$ of min $\boldsymbol{c}^T x : \boldsymbol{A}x \leq \boldsymbol{b}$
  - The Gerlach Theorem: $A_c x - A_\Delta |x| \leq \overline{b}$
  - **Orthant decomposition:** Solve an LP in each orthant and choose the overall minimum (or $-\infty$)

  $\Rightarrow$ For each $s \in \{\pm 1\}^n$ solve:

$$\begin{array}{ll} \text{minimize} & (c_c - D_s c_\Delta)^T x \\ \text{subject to} & (A_c - A_\Delta D_s)x \leq \overline{b} \\ & D_s x \geq 0 \end{array}$$

the diagonal matrix for vector s

The Hard Problems: A Closer Look

For each $s \in \{\pm 1\}^n$ solve:

$$\left.\begin{array}{ll} \text{minimize} & (c_c - D_s c_{\boldsymbol{\Delta}})^T x \\ \text{subject to} & (A_c - A_{\boldsymbol{\Delta}} D_s) x \leq \overline{b} \\ & D_s x \geq 0 \end{array}\right\} \quad 2^n \text{ linear programs}$$

- If k of the variables are non-negative or non-positive, fix the corresponding signs in $s \Rightarrow 2^{n-k}$ linear programs
- If uncertainty only affects l columns, use vectors $s \in \{\pm 1\}^l$ for the corresponding l variables $\Rightarrow 2^l$ linear programs

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- If uncertainty only affects l columns, use vectors $s \in \{\pm 1\}^l$ for the corresponding l variables $\Rightarrow 2^l$ linear programs
- What if the coefficient matrix $\boldsymbol{A}$ is fixed?

Intermezzo: A Matrix Norm

Consider the matrix norm

$$\|A\|_{\infty,1} = \max_{\|x\|_\infty = 1} \|Ax\|_1,$$

where $\|x\|_\infty = \max_i |x_i|$ and $\|x\|_1 = \sum_i |x_i|$.

Theorem (Rohn, 1996)

Deciding whether $\|A\|_{\infty,1} \geq 1$ is NP-hard on the class of positive definite rational matrices.

A Hard Special Case

Theorem (Rohn, 1997)

Deciding whether $\bar{f}(A, \boldsymbol{b}, c) \geq 1$ holds is NP-hard for problems of type $\min c^T x : Ax = \boldsymbol{b},\ x \geq 0$.

Proof idea:

$$\begin{aligned} \text{minimize} \quad & e^T x_1 + e^T x_2 \\ \text{subject to} \quad & A^{-1}x_1 - A^{-1}x_2 = [-e, e] \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\rightarrow \bar{f} = \|A\|_{\infty,1}$$

Feasibility, Optimality and Boundedness

- **Weak infeasibility:** Is there a scenario with no feasible solutions? $\Rightarrow \overline{f} = \infty$
- **Strong optimality:** Does every scenario have an optimal solution? $\Rightarrow \underline{f}, \overline{f}$ are finite
- **Weak unboundedness:** Is there a scenario with an unbounded objective? $\Rightarrow \underline{f} = -\infty$

Theorem (Rohn, 1981; Rohn & Kreslová, 1994)

- The system $\boldsymbol{A}x = \boldsymbol{b}, x \geq 0$ is strongly feasible iff for each $s \in \{\pm 1\}^m$ the system $(A_c - D_s A_\Delta)x = b_c + D_s b_\Delta, x \geq 0$ is feasible.
- The system $\boldsymbol{A}x \leq \boldsymbol{b}$ is strongly feasible if and only if the system $\overline{A}x_1 - \underline{A}x_2 \leq \underline{b}, x_1 \geq 0, x_2 \geq 0$ is feasible.

(In)approximability Properties

Theorem (Rohn, 2000)

- For every $\delta > 0$, computing a rational number that is δ-close to $\|A\|_{\infty,1}$ is NP-hard.
- If $P \neq NP$, then there is no polynomial-time algorithm, which for each non-negative positive definite rational matrix $A \in \mathbb{R}^{n \times n}$ computes a rational approximation r to $\|A\|_{\infty,1}$ satisfying

$$\left| \frac{r - \|A\|_{\infty,1}}{\|A\|_{\infty,1}} \right| \leq \frac{1}{4n^2}.$$

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Also holds for the best optimal value!

Approximating the Best Optimal Value

Goal: Find an upper bound $\underline{f}^U$ and a lower bound $\underline{f}^L$ on the best optimal value $\underline{f}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ of the problem

$$\text{minimize } \boldsymbol{c}^T x \text{ subject to } \boldsymbol{A}x \leq \boldsymbol{b}.$$

- Upper bound: optimal value of any scenario in $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$
- Lower bound: optimal value of a relaxed problem

Lower Bound on the Best Case Value

$$\min_{A \in \boldsymbol{A}, c \in \boldsymbol{c}} \left\{ \min_{x \in \mathbb{R}^n} c^T x \text{ subject to } Ax \leq \overline{b} \right\}$$

Lower Bound on the Best Case Value

$$\min c^T x \text{ subject to } Ax \leq \overline{b},\ \underline{c} \leq c \leq \overline{c},\ \underline{A} \leq A \leq \overline{A}$$

McCormick Envelopes

$$f(x, y) = xy,\ \underline{x} \leq x \leq \overline{x},\ \underline{y} \leq y \leq \overline{y}$$

$$\begin{aligned} w &\geq \underline{x}y + x\underline{y} - \underline{x}\underline{y}, \quad & w &\geq \overline{x}y + x\overline{y} - \overline{x}\overline{y}, \\ w &\leq \overline{x}y + x\underline{y} - \overline{x}\underline{y}, \quad & w &\leq x\overline{y} + \underline{x}y - \underline{x}\overline{y} \end{aligned}$$

Lower Bound: McCormick Envelopes Relaxation

$$
\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{n} w_i \\
\text{subject to} \quad & Ax \leq \overline{b} \\
& \underline{c} \leq c \leq \overline{c} \\
& w_i \geq \underline{c_i} x_i + c_i \underline{x_i} - \underline{c_i}\underline{x_i}, \quad && i \in \{1, \ldots, n\} \\
& w_i \geq \overline{c_i} x_i + c_i \overline{x_i} - \overline{c_i}\overline{x_i}, \quad && i \in \{1, \ldots, n\} \\
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& w_i \leq c_i \overline{x_i} + \underline{c_i} x_i - \underline{c_i}\overline{x_i}, \quad && i \in \{1, \ldots, n\}
\end{aligned}
$$

+ constraints for A

Upper Bound on the Best Case Value

Algorithm 3 Upper bound $\underline{f}^U$ on $\underline{f}$

```
compute f* := f(A^c, b, c^c) and let x* be the corresponding optimal solution
repeat
    put f^U := f*
    put s := sgn(x*)
    compute the optimal value f* and the optimal solution x* to (8)
until f^s ≥ f^U or s = sgn(x*)
return f^U := min{f^U, f*}
```

- compute $\underline{f}^* := f(A^c, b, c^c)$ and let x^* be the corresponding optimal solution
- **repeat**
  - put $\underline{f}^U := \underline{f}^*$
  - put $s := \mathrm{sgn}(x^*)$
  - compute the optimal value $\underline{f}^*$ and the optimal solution x^* to (8)
- **until** $\underline{f}^s \geq \underline{f}^U$ **or** $s = \mathrm{sgn}(x^*)$
- **return** $\underline{f}^U := \min\{\underline{f}^U, \underline{f}^*\}$

and the initial bound $\underline{f}^U := f(A^c, b, c^c)$. Then, we run an iterative local improvement method to find a scenario with as small as possible optimal value.

Put $s := \mathrm{sgn}(x^*)$. The best case optimal value for the feasible set restricted to the orthant $D_s x \geq 0$, is calculated by the linear program (6). This motivates us to choose the following scenario of (3) as a promising one for achieving the lowest optimal value.

$$\underline{f}^s := \min (c^c - D_s c^\Delta)^T x \quad \text{subject to} \quad (A^c - A^\Delta D_s)x \leq b. \tag{8}$$

Milan Hladík. On approximation of the best case optimal value in interval linear programming (2014)

Optimal Value Range: Example

Example (Hladík, 2014)

$$\text{minimize } [2,3]x_1 + [6,7]x_2$$

$$\begin{pmatrix} -[4,5] & -[2,3] \\ [4,5] & -[1,2] \\ [2,3] & [5,6] \end{pmatrix} x \leq \begin{pmatrix} -[11,12] \\ [26,28] \\ [43,45] \end{pmatrix}$$

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Worst optimal value:

$$\text{maximize } \underline{b}^T y \text{ subject to } \overline{A}^T y \leq \overline{c},\ \underline{A}^T y \geq \underline{c},\ y \leq 0$$

$$\rightarrow \bar{f} = 1.8261$$

Optimal Value Range: Example (cont.)

Best optimal value:

- Upper bound:
  1. Solve the scenario with $A_c, \overline{b}, c_c$: $x^* = (4.8056, -4.2500)$, $f(x^*) = -15.6111$.
  2. Modify the coefficients using $s = (1, -1)$ and solve the corresponding LP: $x^s = (5.1538, -7.3846)$, $f(x^s) = -41.3846$.
  3. Sign vector s is the same, $\underline{f}^U = -41.3846$.

     exact best value
- Lower bound:
  1. Compute an interval envelope of the feasible set: $x \in [-7.3, 9.6] \times [-7.4, 13.4]$.
  2. Replace the bilinear terms with the McCormick envelope and solve the obtained LP: $\underline{f}^L = -44.4189$.

Optimal Value Range: Example (cont.)

Best optimal value:

- Upper bound:
  1. Solve the scenario with $A_c, \overline{b}, c_c$: $x^* = (4.8056, -4.2500)$, $f(x^*) = -15.6111$.
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  3. Sign vector s is the same, $\underline{f}^U = -41.3846$.

  exact best value

- Lower bound:
  1. Compute an interval envelope of the feasible set: $x \in [-8, 10] \times [-8, 15]$.
  2. Replace the bilinear terms with the McCormick envelope and solve the obtained LP: $\underline{f}^L = -48.3414$.

Conclusion

We discussed the problem of computing the optimal value range in interval linear programming...

- For programs of type $\boldsymbol{A}x \leq \boldsymbol{b}, x \geq 0$, we can compute the optimal value range exactly and quickly.
- For programs of type $\boldsymbol{A}x \leq \boldsymbol{b}$ (or $\boldsymbol{A}x = \boldsymbol{b}, x \geq 0$), one of the bounds is difficult (time-consuming) to compute exactly, even with a fixed matrix A. So, we approximate it!

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Thank you for your attention!